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In 2010 all ECB publications feature a motif taken from the €500 banknote.

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Abstract

We show how to use a simple perturbation method to solve non-linear rational expectation models. Drawing from the applied mathematics literature we propose a method consisting of series expansions of the non-linear system around a known solution. The variables are represented in terms of their orders of approximation with respect to a perturbation parameter. The final solution, therefore, is the sum of the different orders.

This approach links to formal arguments the idea that each order of approximation is solved recursively taking as given the lower order of approximation. Therefore, this method is not subject to the ambiguity concerning the order of the variables in the resulting state-space representation as, for example, has been discussed by Kim et al. (2008).

Provided that the model is locally stable, the approximation technique discussed in this paper delivers stable solutions at any order of approximation.

JEL *classification: C63; E0*

Keywords: Solving dynamic stochastic general equilibrium models; Perturbation methods; Series expansions; Non-linear difference equations.

Non Technical Summary

This paper addresses an important open issue in the economic literature: how to solve non-linear dynamic models. Higher-order solutions (i.e. not simply linear) have been found crucial in order to address a number of important questions, like for example policy questions or financial issues, as only to higher-order of approximation we can meaningfully evaluate the role of risk and uncertainly.

A number of recent contributions to this literature agree that perturbation methods, and in particular approximations by series expansions, can result in useful characterizations of the solutions to dynamic stochastic general equilibrium models (DSGE), as for example discussed in Judd (1998). Nevertheless, not all these contributions agree on the correct representation of these solutions and, hence, on how to compute impulse response functions and other moments. Kim et al. (2008) show that the typical representation of the state-space model consists of a non-linear polynomial in the state variables. They show that this polynomial will generally have multiple steady states and could yield unbounded solutions. These authors propose a procedure to circumvent this problem ("pruning"). Schmitt-Grohé and Uribe (2004) explain in detail how the perturbation method described by Judd (1998) should be applied to DSGE models. In their paper the state-space representation of the solution does not explicitly distinguish between first and second order variables: the second order expression is represented as a quadratic polynomial in the state variables. Nevertheless, in the accompanying computer code for simulations they use the "pruned" solution.[1] Collard and Juillard (2001) propose a perturbation approximation that does not explicitly differentiate the order of the variables. Gomme and Klein (2010) provide an alternative solution method for the coefficient matrices of

[1] I thank Martin Uribe for pointing this out to me.

second order approximation. In their paper the authors do not distinguish the order of the variables in the second order expansion. Widely used computer softwares that solve DSGE models to higher order, like Dynare (and Dynare++) (Juillard, 1996) or PerturbationAIM (Swanson et al., 2006) produce state space representations that are not "pruned".[2]

Part of the literature therefore offers representations of the solution that are intrinsically non-linear and that, therefore, can generate global dynamics that differ considerably from the local behavior of the model. These global dynamics do not (necessarily) reflect the global properties of the original model and, hence, are not informative about the behavior of the economy under large shocks. Neverheless, their presence constitutes a big problem in solving medium-to-large scale DSGE models, leaving us with limited practical tools to address interesting questions (e.g. the size of risk-premia in medium-scale DSGE models used at central banks).

Lombardo and Sutherland (2007), on the contrary, suggest a solution to DSGE models that does not require any ad-hoc alteration of the state-space representation: their solution is recursively linear and hence preserves the stationary properties of the firs-order approximation, despite being accurate to second order. Their work, nevertheless, does not provide fully formal arguments to prove the mathematical consistency of their solution with the proposed series expansion methods and so the link to the rest of the literature remains only suggestive.

The present paper fills the gap between the alternative approaches with the intent to avoid ambiguities regarding the implementation of the proposed solution. In particular we borrow from the applied-mathematics literature on perturbation methods

[2]The user manuals for these computer programs describe the solution as a non-linear difference equation. At present, Dynare version 4.2 offers the pruning option for second order approximations. This option is not yet available to higher order.

(e.g Holmes, 1995) to provide a formally consistent higher-order approximation to a non-linear model that is recursively linear as in Lombardo and Sutherland (2007). Using the standard neo-classical growth model we show that the solution proposed here, like that of Lombardo and Sutherland (2007) generates stationary dynamics independently of the size of the shocks, contrary to the alternative representations.

1 Introduction

The literature on higher order approximations of dynamic stochastic general equilibrium (DSGE) models is ambiguous about the interpretation of the implied state-space solutions. Following the seminal work of Jin and Judd (2002), Judd (1998) and Judd (2002) a number of papers has contributed to the literature on higher order approximations by suggesting alternative techniques to solve and simulate DSGE models. Lombardo and Sutherland (2007) develop a solution technique based on the recursive linearity of the state-space representation. In that paper the authors resort to an "order" argument to maintain that cross-products of variables in higher order expressions must be computed using lower order terms. Schmitt-Grohé and Uribe (2004) refer more closely to the work of Judd and present a state-space solution that does not appear to exploit the recursive-linearity.[3] Finally Kim et al. (2008) discuss a solution technique similar to Schmitt-Grohé and Uribe (2004) and explicitly discuss the problem of using the state-space representation for simulations. Their paper, in particular, emphasizes that the state-space representation à la Schmitt-Grohé and Uribe (2004), being a polynomial of higher order, could generate explosive solutions: the true solution is not interpreted as recursively linear as in Lombardo and Sutherland (2007). Kim et al. (2008) suggest to amend the second order state-space representation by replacing cross-products of variables with cross-products of variables obtained from first order solutions: a procedure they name "pruning".[4] Collard and Juillard (2001) propose a perturbation approximation that does not explicitly differentiate the order of the variables. Gomme and Klein (2010) provide an alterna-

[3] The authors show the state-space in non-linear form but don't discuss the potential issues related to its non-linear structure. Nevertheless, the accompanying computer code posted by the authors on the web, "simu_2nd.m", applies the "pruning" procedure. I thank Martin Uribe for having pointed out this fact to me.

[4] The "pruned" solution is identical to that derived in Lombardo and Sutherland (2007).

tive solution method for the coefficient matrices of second order approximation. In their paper the authors do not distinguish the order of the variables in the second order expansion.

Den Haan and de Wind (2010) compare perturbation methods with projection methods used for approximating and solving DSGE models. These authors emphasize how perturbation methods can easily lead to explosive solutions, making the approach problematic. They also discuss the "pruning" procedure proposed by Kim et al. (2008) stressing the fact that this procedure is a work-around to an intrinsic problem of perturbation methods. They document "... that this procedure is quite distortive" (den Haan and de Wind, 2010, p. 22).

In this paper we borrow from the applied-mathematics literature to show that approximating DSGE models using the method of series expansions naturally generates solutions that are recursively linear. Or, as explained by Berglund (2001, p. 3)"... at each order we only need to solve [a] linear equation, where the [non-linear] term depends only on previously computed quantities."

Compared with the existing literature the approach suggested here differs in the way the conjectured solution is represented.We don't discuss the origins of these two alternative representations. The approach followed here is thoroughly discussed in e.g. Holmes (1995), Bush (1992) and Hinch (1991). The main focus of the paper is to provide a solution method to DSGE models that can avoid the ambiguity that surrounds the existing economics literature on higher order approximations.

As a result of the recursive linearity, similarly to Lombardo and Sutherland (2007), our method implies that the solution of the DSGE model is stationary (non-explosive) to any order of approximation as long as the model is locally stable. In this sense, our solution is not subject to the criticism raised by den Haan and de Wind (2010).

The rest of the paper is organized as follows. Section 2 provides an overview of the technique. Section 3 applies the technique to a simple non-linear DSGE model and compares the alternative methods. Section 4 concludes.

2 The method of series expansions

In this paper we are interested in solving problems of the general form

$$\mathbb{E}_t F\left(z_{t+1}, z_t, z_{t-1}, \varepsilon_t\right) = 0$$

where F is a system of non-linear stochastic difference equations deriving from first order conditions of agents' problems, resource constraints and market clearing conditions, $\mathbb{E}_t$ is the mathematical expectation operator, ε_t is a vector of exogenous stochastic forcing processes with given low of motion and z_t is a vector of endogenous variables. In this section, though, we will first describe the approximation method by using a very simplified version of the general case. This version would indeed not need approximations as the value of the variable at each point in time can be easily traced starting from given initial conditions. In the next section we will return to the more general case, showing an application to the neo-classical growth model.

From Holmes (1995) and Berglund (2001) we see that in order to approximate the (scalar) model

$$y_t = f\left(\varepsilon, y_{t-1}\right) \tag{1}$$

we can proceed as follows.[5] First, "we assume" (Holmes, 1995, p. 27) a solution of

[5] Obviously, applying this method requires that the solution admits series expansion in the perturbation parameter and that the function f is analytic. In particular the resulting Jacobian of the system of equations must be non-singular. When this condition is not satisfied singular-perturbation methods can be applied. Singular perturbation methods have been applied to portfolio problems by Judd and Guu (2001).

the form (i.e. a series expansion in ε)[6]

$$y_t = y_t^{(0)} + \varepsilon^{\alpha} y_t^{(1)} + \varepsilon^{2\alpha} y_t^{(2)} \tag{2}$$

where $^{(x)}$ denotes the order x of the variable, i.e. $y_t^{(0)} = \mathcal{O}(\varepsilon^0)$ etc. Notice that $y_t^{(x)}$ is assumed not to depend on the perturbation parameter. It is important to notice that this representation is identical to that discussed in Judd (1998, p. 456-457). There it is clear that the terms in $y_t^{(x)}$ are "derivatives of $[y_t(\varepsilon)]$ with respect to ε when $\varepsilon = 0$."

For simplicity we consider the case of $\alpha = 1$ although this might not be the best choice in general (see Holmes, 1995 and Judd, 1998, p. 516).

Then, we take an expansion of $f(\cdot)$ around $\varepsilon = 0$ (assume that $y_t^{(0)} = y_0$)

$$f(\varepsilon, y_t) \approx f(0, y_0) + f_{\varepsilon}\varepsilon + f_y(y_t - y_0) + \frac{1}{2}\left(f_{\varepsilon\varepsilon}\varepsilon^2 + f_{yy}(y_t - y_0)^2\right) + \mathcal{O}\left(\varepsilon^3\right) \tag{3}$$

where for simplicity we have assumed that $f_{\varepsilon y} = 0$.

Replace the conjectured solution and the approximation in the original problem (1),[7]

$$\begin{aligned} y_t^{(0)} + \varepsilon y_t^{(1)} + \varepsilon^2 y_t^{(2)} + \ldots \; = \\ f(0, y_0) + f_{\varepsilon}\varepsilon + f_y\left(\varepsilon y_{t-1}^{(1)} + \varepsilon^2 y_{t-1}^{(2)} + \ldots\right) \\ + \frac{1}{2}\left(f_{\varepsilon\varepsilon}\varepsilon^2 + f_{yy}\left(\varepsilon y_{t-1}^{(1)} + \varepsilon^2 y_{t-1}^{(2)} + \ldots\right)^2\right) + \mathcal{O}\left(\varepsilon^3\right). \end{aligned} \tag{4}$$

"By equating like powers" (Holmes, 1995, p. 27) we obtain that the zero order is

$$y_t^{(0)} = f(0, y_0),$$

[6] Quoting Holmes: "[This assumption] is nothing more than an educated guess. The motivation for making this assumption comes from the observation that in expanding functions one usually ends up using Taylor's theorem and [our guess] is simply a reflection of that type of expansions."

[7] Notice that we could have first substituted the guess into the function and then expanded with respect to ε. The result would have been the same.

that the first order is

$$y_t^{(1)} = f_y y_{t-1}^{(1)} + f_\varepsilon + \mathcal{O}\left(\varepsilon^2\right),$$

and that the second order is

$$y_t^{(2)} = f_y y_{t-1}^{(2)} + \frac{1}{2}\left(f_{\varepsilon\varepsilon} + f_{yy}\left(y_{t-1}^{(1)}\right)^2\right) + \mathcal{O}\left(\varepsilon^3\right).$$

Therefore

$$y_t - y_0 \approx \varepsilon\left(f_y y_{t-1}^{(1)} + f_\varepsilon\right) + \varepsilon^2\left(f_y y_{t-1}^{(2)} + \frac{1}{2}\left(f_{\varepsilon\varepsilon} + f_{yy}\left(y_{t-1}^{(1)}\right)^2\right)\right).$$

Notice that there are terms of order higher than 2 in equation (4). These belong to the higher order terms in the residual, i.e. they belong in $\mathcal{O}\left(\varepsilon^3\right)$.

Definition of solution of the system of stochastic difference equations

Consider the non-autonomous linear system in the vector y_t and forcing process f_t,

$$y_t = A\, y_{t-1} + f_t,$$

where A is a matrix of coefficients with all eigenvalues lying within the unit circle.

We define the solution to this system as its moving average representation, e.g. as a function of initial conditions (y_0) and the forcing process, i.e.[8]

$$y_t = A^t y_0 + \sum_{i=0}^{t-1} A^i f_{t-i}.$$

In the rest of the paper, following most of the related literature, we will refer to the "solution" of the model as the state-space representation of the problem. Obviously, in order to obtain the solution, we need to perform a further step. This step sheds

[8] See Ljungqvist and Sargent (2000, p. 11). More in general the solution of a non-autonomous difference equation would be a function of a constant and the infinite history of the forcing process (Kelley and Peterson, 2001). See also Judd (1998, p. 338).

light on the differences between the perturbation method proposed here and some representations of the higher order solutions discussed in the literature, as we will make clear later in the paper.

3 An example: The Neo-Classical Growth Model

In this section we apply the method of series expansion to the Neo-Classical Growth Model. This model has also been used by Judd (1998), Lombardo and Sutherland (2007) and Schmitt-Grohé and Uribe (2004) to show how to apply their approximation techniques.

Consider the following rational expectation model, consisting of an Euler consumption (c) equation, a capital (k) accumulation equation and an i.i.d. process for the (log) of the productivity shock (ε) with zero mean and variance normalized to one.[9] That is

$$c_t^{-\gamma} = \alpha\beta\mathbb{E}_t\left[e^{\sigma\varepsilon_{t+1}}k_{t+1}^{\alpha-1}c_{t+1}^{-\gamma}\right] \tag{5}$$

$$k_{t+1} = e^{\sigma\varepsilon_t}k_t^{\alpha} - c_t \tag{6}$$

where $\sigma > 0$, $\alpha \in (0,1)$, $\gamma > 0$ and $\beta \in (0,1)$.

Notice that if $\sigma = 0$, the model is deterministic and has a closed form solution $k_0 = (\alpha\beta)^{\frac{1}{1-\alpha}}$ and $c_0 = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}$.

The first step of the series expansion method consists of assuming that the solu-

[9] For the approximation to be valid, the distribution of the shock must be such that the system remains within its radius of convergence. Here we leave this issue in the background referring the reader to Jin and Judd (2002) and Kim et al. (2008) for a discussion of this issue. For simplicity we assume zero persistence in the productivity shock and full depreciation in the capital stock. These assumptions are also made in Lombardo and Sutherland (2007) and Schmitt-Grohé and Uribe (2004).

tion can be expressed in terms of a series expansion

$$c_t \approx c_0 + \sigma c_t^{(1)} + \sigma^2 c_t^{(2)} \tag{7}$$

and

$$k_t \approx k_0 + \sigma k_t^{(1)} + \sigma^2 k_t^{(2)} \tag{8}$$

Take a second order expansion of equations (5) and (6) around $\sigma = 0$, dropping the expectation operator for notational convenience (implicitly associated to $t+1$ variables) we have

$$\begin{aligned}\mathcal{O}\left(\varepsilon^3\right) = \gamma c_0^{-\gamma-1}\hat{c}_t + \alpha\beta k_0^{\alpha-1}c_0^{-\gamma}\left[\sigma\varepsilon_{t+1} + k_0^{-1}\left(\alpha-1\right)\hat{k}_{t+1} - \gamma c_0^{-1}\hat{c}_{t+1}\right] \quad + \\ -\frac{1}{2}\gamma\left(\gamma+1\right)c_0^{-\gamma-2}\hat{c}_t^2 + \frac{1}{2}\alpha\beta k_0^{\alpha-1}c_0^{-\gamma}\left[\sigma^2\varepsilon_{t+1}^2 + \gamma\left(\gamma+1\right)c_0^{-2}\hat{c}_{t+1}^2 + \left(\alpha-1\right)\left(\alpha-2\right)k_0^{-2}\hat{k}_{t+1}^2\right] \quad + \\ +\alpha\beta k_0^{\alpha-1}c_0^{-\gamma}\left[k_0^{-1}\left(\alpha-1\right)\sigma\varepsilon_{t+1}\hat{k}_{t+1} - \gamma c_0^{-1}\sigma\varepsilon_{t+1}\hat{c}_{t+1} - \gamma\left(\alpha-1\right)k_0^{-1}c_0^{-1}\hat{c}_{t+1}\hat{k}_{t+1}\right]\end{aligned}$$

and

$$\begin{aligned}\mathcal{O}\left(\varepsilon^3\right) = -\hat{k}_{t+1} + k_0^{\alpha}\sigma\varepsilon_t + \alpha k_0^{\alpha-1}\hat{k}_t - \hat{c}_t \quad + \\ \sigma\alpha k_0^{\alpha-1}\varepsilon_t\hat{k}_t + \frac{1}{2}k_0^{\alpha}\sigma^2\varepsilon_t^2 + \frac{1}{2}\alpha\left(\alpha-1\right)k_0^{\alpha-2}\hat{k}_t^2\end{aligned}$$

Define $z_t = \left[\hat{c}_t\,\hat{k}_t\right]'$. Then can rewrite in matrix notation

$$\begin{aligned}\mathcal{O}\left(\varepsilon^3\right) = A_2 z_{t+1} + A_1 z_t + \sigma C_0\varepsilon_t + \\ +B_2 w_{t+1} + B_1 w_t + \sigma D_2 z_{t+1}\varepsilon_{t+1} + \sigma D_1 z_t\varepsilon_t + \sigma^2 C_1\varepsilon_t^2 \quad +\sigma^2 C_2\varepsilon_{t+1}^2\end{aligned} \tag{9}$$

where[10]

[10] We are using the *vech* operator to eliminate repeated terms in kronecker products of identical vectors. For approximations of order larger than 2, appropriate elimination matrices can be applied to eliminate repeated terms in n-th tensor powers.

$$w_t = vech\left(z_t z_t'\right) = \left[\begin{array}{ccc} \hat{c}_t^2 & \left(\hat{c}_t \hat{k}_t\right) & \hat{k}_t^2 \end{array}\right]'$$

$$A_2 = \left[\begin{array}{cc} -\gamma\alpha\beta k_0^{\alpha-1} c_0^{-\gamma-1} & \alpha\beta\left(\alpha-1\right) k_0^{\alpha-2} c_0^{-\gamma} \\ 0 & -1 \end{array}\right]$$

$$A_1 = \left[\begin{array}{cc} \gamma c_0^{-\gamma-1} & 0 \\ -1 & \alpha k_0^{\alpha-1} \end{array}\right]$$

$$C_0 = \left[\begin{array}{c} 0 \\ k_0^{\alpha} \end{array}\right]$$

$$C_1 = \frac{1}{2}\left[\begin{array}{c} 0 \\ k_0^{\alpha} \end{array}\right]$$

$$C_2 = \frac{1}{2}\left[\begin{array}{c} \alpha\beta k_0^{\alpha-1} c_0^{-\gamma} \\ 0 \end{array}\right]$$

$$B_2 = \frac{1}{2}\left[\begin{array}{ccc} \gamma\left(\gamma+1\right)\alpha\beta k_0^{\alpha-1} c_0^{-\gamma-2} & -2\alpha\beta k_0^{\alpha-2} c_0^{-\gamma-1}\gamma\left(\alpha-1\right) & \left(\alpha-1\right)\left(\alpha-2\right)\alpha\beta k_0^{\alpha-3} c_0^{-\gamma} \\ 0 & 0 & 0 \end{array}\right]$$

$$B_1 = \frac{1}{2}\left[\begin{array}{ccc} -\gamma\left(\gamma+1\right) c_0^{-\gamma-2} & 0 & 0 \\ 0 & 0 & \alpha\left(\alpha-1\right) k_0^{\alpha-2} \end{array}\right]$$

$$D_2 = \left[\begin{array}{cc} -\gamma\alpha\beta k_0^{\alpha-1} c_0^{-\gamma-1} & \alpha\beta k_0^{\alpha-2} c_0^{-\gamma}\left(\alpha-1\right) \\ 0 & 0 \end{array}\right]$$

$$D_1 = \left[\begin{array}{cc} 0 & 0 \\ 0 & \alpha k_0^{\alpha-1} \end{array}\right]$$

We can also rewrite our assumption concerning the solution in matrix notation[11]

$$z_t = \sigma z_t^{(1)} + \sigma^2 z_t^{(2)} \tag{10}$$

Notice that

$$z_t z_t' = \sigma^2 z_t^{(1)} z_t^{(1)\prime} + \sigma^4 z_t^{(2)} z_t^{(2)\prime} + \sigma^3 z_t^{(1)} z_t^{(2)\prime} + \sigma^3 z_t^{(2)} z_t^{(1)\prime} \tag{11}$$

Since we are interested in solution up to order 2, we must drop the higher terms in σ.[12] Therefore

$$\sigma^2 \hat{w}_t = \sigma^2 vech\left(z_t^{(1)} z_t^{(1)\prime}\right).$$

Replacing equations (10) and (11) into (9) we obtain

$$\begin{aligned} A_2\left(\sigma z_{t+1}^{(1)} + \sigma^2 z_{t+1}^{(2)}\right) + A_1\left(\sigma z_t^{(1)} + \sigma^2 z_t^{(2)}\right) + \sigma C_0 \varepsilon_t + \\ + B_2 \sigma^2 \hat{w}_{t+1} + B_1 \sigma^2 \hat{w}_t + \\ \sigma D_2\left(\sigma z_t^{(1)} + \sigma^2 z_t^{(2)}\right)\varepsilon_t + \sigma D_1\left(\sigma z_{t+1}^{(1)} + \sigma^2 z_{t+1}^{(2)}\right)\varepsilon_{t+1} \\ + \sigma^2 C_1 \varepsilon_t^2 + \sigma^2 C_2 \varepsilon_{t+1}^2 = 0. \end{aligned}$$

By equating like powers we have

$$\sigma : A_2 z_{t+1}^{(1)} + A_1 z_t^{(1)} + C_0 \varepsilon_t = 0 \tag{12}$$

$$\begin{aligned} \sigma^2 : A_2 z_{t+1}^{(2)} + A_1 z_t^{(2)} + B_2 vech\left(z_{t+1}^{(1)} z_{t+1}^{(1)\prime}\right) + B_1 vech\left(z_t^{(1)} z_t^{(1)\prime}\right) + \\ + D_2 z_{t+1}^{(1)} \varepsilon_{t+1} + D_1 z_t^{(1)} \varepsilon_t + C_2 \varepsilon_{t+1}^2 + C_1 \varepsilon_t^2 = 0. \end{aligned} \tag{13}$$

[11] Notice that we have reformulated the assumption concerning the solution: now we are expanding the deviations from the steady-state.

[12] As explained in the previous section, these higher order terms will match terms in the approximation residual.

Both these equations can be solved (recursively) using solution techniques for linear RE as discussed in Lombardo and Sutherland (2007) or the method of undetermined coefficients.[13]

Notice that provided that system (12) has a stationary solution – i.e. that the number of eigenvalues λ of the matrix pencil $A_1 - \lambda A_2$ that lie inside the unit circle is equal to the number of predetermined variables (Blanchard and Kahn, 1980 and Klein, 2000) – the system (13) will also be stationary. The Blanchard-Kahn stationarity conditions for this second-order linear system are identical to those of its companion first-order system: if the latter is stable the former is stable too. This holds true for any order of approximation.

The final solution is obtained by solving for $z_t^{(1)}$ from (12) and for $z_t^{(2)}$ from (13) and replacing the solutions into (10).

It is crucial to notice that the matrix of coefficients of the resulting state-space solution are the same that would be found with the alternative methods discussed in this paper. In particular, notice that these alternative approaches, with the exception of Lombardo and Sutherland (2007), postulate, for the vector of control variables y_t and state variables x_t, a solution of the form

$$y_t = g\left(x_t; \sigma\right),$$

and

$$x_{t+1} = h\left(x_t; \sigma\right) + \sigma\eta\varepsilon_{t+1},$$

where η is the variance-covariance matrix of the innovations and σ remains the scalar perturbation parameter. By taking an n-th order Taylor expansion of these equations, replacing each variable with its expansion to n-th order and matching powers, we end

[13] Any solver for linear rational expectation models can be used, e.g. Christiano (2002), King and Watson (2002), Klein (2000), Uhlig (1999) etc.

up with a recursively linear system of difference equations with unknown coefficients. By applying the method of undetermined coefficients we obtain the same state-space solution obtained with the direct method used by Lombardo and Sutherland (2007).

It is important to notice that in general there is no formal need to postulate the solution in terms of a policy function when we use the perturbation method proposed in this paper. Only if we decide to use the method of undetermined coefficients we have to take a stand on the unknown policy function.

3.1 Pruning and higher order expansions

A word of caution is in order concerning the application of the "pruning" procedure to orders of approximation higher than two.[14] As "pruning" is an ad-hoc procedure intended to eliminate cross-products of endogenous variables in the second order expansion, it does not offer guidance on how to amend higher order approximations. For example, consider an approximation to third order. This would generally contain terms of the form $A\,x_t + F\mathbb{E}_t\eta^2\varepsilon^2_{t+1}x^{(1)}_t + \ldots$.[15] Without loss of generality, we can continue to assume that $\mathbb{E}_t(\varepsilon^2_{t+1}) = 1$. Then, if the order of the variables is not taken into account, one would write $(A + F\eta^2)\ x_t$. Thus the coefficient matrix on the linear term of the approximation would conflate matrices applying to variables of different order.[16] Our paper makes clear that amending higher order solutions obtained using alternative approaches (i.e. "pruning") might require the cumbersome procedure of reconstructing the appropriate coefficient matrices.

In the rest of this paper we focus on the second order expansion, for which Dynare output can be safely used to generate pruned solutions.

[14] This section relies heavily on the excellent research assistance by Szabolcs Deak.

[15] The same argument applies with some modifications if terms like $\varepsilon^2_t x^{(1)}_{t-1}$ enter the expansion.

[16] This is the case with Dynare++.

3.2 Comparing the solutions

In this section we compare the solution of the neo-classical growth model obtained with the method suggested in this paper with the solution obtained with the method suggested by Lombardo and Sutherland (2007) and with the perturbation without pruning (NP henceforth).[17]

We use DYNARE to generate the solution of the first and second order approximation (Juillard, 1996).

The solution can be represented in the following compact form

$$\hat{z}_t^{(1)} = A\hat{z}_{t-1}^{(1)} + B\varepsilon_t \tag{14}$$

$$\hat{z}_t^{(2)} = \frac{1}{2}\Delta + A\hat{z}_{t-1}^{(2)} + \frac{1}{2}C\left(z_{t-1}^{(1)} \otimes z_{t-1}^{(1)}\right) + \frac{1}{2}D\varepsilon_t^2 + E\hat{z}_{t-1}^{(1)}\varepsilon_t \tag{15}$$

and

$$\hat{z}_t = \sigma\hat{z}_t^{(1)} + \sigma^2\hat{z}_t^{(2)}, \tag{16}$$

where for the neo-classical growth model we have[18]

$$\hat{z}_t = \left[\hat{k}_{t+1}\hat{c}_t\right]'$$

$$\Delta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0.1 & 0 \\ 0.953 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.0731 \\ 0.697 \end{pmatrix}$$

[17] Notice that we use a level-expansion as opposed to log-expansion.
[18] We assume that $\gamma = 1$, $\alpha = 0.1$ and $\beta = 0.95$.

$$C = \begin{pmatrix} -1.23 & 0 & 0 & 0 \\ -11.7 & 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0.0731 \\ 0.697 \end{pmatrix}$$

$$E = \begin{pmatrix} 0.1 \\ 0.953 \end{pmatrix}$$

A comment is in order concerning the non-linearity captured by the solution given in equations (14) to (16). To emphasize the recursive (or conditional) linearity we have displayed these equations in an ordered way: for a given innovation ε_t and the state of the economy at time $t-1$ we can generate values for z_t by simply evaluating each equation in the given order. At each subsequent step, the non-linearity is captured by the interaction terms. This implies, for example, that if ε_t is Gaussian, $z_t^{(2)}$ and z_t are non-Gaussian.[19]

It should also be clear from the solution that it does not amount to the solution discussed in the current literature on approximating DSGE models. In particular, Lombardo and Sutherland (2007) and Kim et al. (2008) would have a linear term in the innovation ($B\varepsilon_t$) also in the second order equation. Furthermore, the (approximate) solution discussed here consists of the weighted average of first- and second-order variables, while both the other methods would treat the solution for $z_t^{(2)}$ as the (approximate) solution for z_t. This latter point implies that further reducing our solution to a function of initial conditions and exogenous processes (i.e. solving the system of difference equations) would yield the term $A^t \hat{z}_0^{(1)}$, which would be absent

[19] Notice that these variables are not even conditionally Gaussian, due to the appearing of powers of the innovations.

in the alternative solutions discussed here.[20] Using the non-pruned solution would not give the same closed form solution, further stressing the deep difference between the two approaches.

Despite these differences all these alternative approaches yield very similar results for small shocks to second order of approximation.

Figures 1 and 2 show the result of random simulations of three alternative specifications of the solution: the solution described in this paper ("TP") the Lombardo and Sutherland (2007) solution ("LS") – equivalent to Kim et al. (2008) with "pruning" – and the solution without pruning ("NP") – equivalent to Kim et al. (2008) without "pruning".

Each figure shows four panels. The first row shows the difference between TP and NP, while the second shows the difference between TP and LS. The columns refer to capital and consumption respectively. Values are reported relative to the non-stochastic steady state.[21] Figure 1 shows the case of "small" standard deviations ($\sigma = 0.1$). It is clear that the three specifications produce very similar results. The TP and LS solutions are virtually identical while NP differs by small amounts. Figure 2 shows the case of "large" standard deviations ($\sigma = 1.5$). Even in this case the TP-LS discrepancies are virtually zero. In contrast, now the NP solution differs markedly from our solution (TP). The difference is due to the fact that the NP solution is non-linear (as opposed to recursively linear) so that for large shocks the dynamics of the variables ceases to be governed by the local stability properties of the process.[22] This fact has motivated the "pruning" fix suggested by Kim et al.

[20] In most applications, though, this term would be zero – when starting from the steady-state (mean of linear process) – or negligible – when considering stochastic simulations of sufficient length.

[21] We ran 5000 iterations dropping the first 500 to plot the graphs.

[22] Further increasing the standard deviation would show even larger discrepancies. Notice that here we are not arguing that with large standard deviations the approximation would be reasonably accurate. The point stressed here is that the divergence in the approximation does not necessarily

(2008). This paper suggests, on the contrary, that the problem can be avoided if the solution is constructed consistently with the series-expansion method suggested in the perturbation literature we have referred to in this paper.

Accuracy The focus of this paper is not about accuracy of the approximations. We discuss an approximation method that is consistent with the series expansion method treated in the perturbation literature. To which extent the approximate solution is accurate will depend on the particular problem. We notice nevertheless that increasing the order of approximation makes the divergence of the non-linearly-recursive representation more likely, potentially reducing the accuracy as we increase the order of approximation.[23] We can easily see this with the neo-classical growth model described above. With the particular parametrization used here it is possible to derive the exact solution in closed form. That is

$$c_t = (1-\alpha\beta)e^{\varepsilon_t}k_t^{\alpha} \tag{17}$$

$$k_{t+1} = \alpha\beta e^{\varepsilon_t}k_t^{\alpha}. \tag{18}$$

We simulated this model and compared the solution to the alternative approximation techniques discussed here.[24] For example assuming a standard deviation of the shock of 0.5 yields the mean of consumption and capital for the different models (in level deviations) reported in Table 1.

reflect global properties of the original model. Importantly, experiments (not reported) with larger models show that diverging paths can be produced with standard errors suggested by estimations and deemed economically reasonable.

[23] For a given point within the radius of convergence, the approximation residual of an analytic function will converge to zero as the order of approximation goes to infinity. Nevertheless, this will not happen monotonically (e.g. the sine function). The inaccuracy generated by non recursively-linear solutions is, once more, related to the recursive non-linearity of the solution.

[24] We used Dynare++ in order to simulate the NP representation to orders higher than 2.

While the approximation error for the three methods is comparable and relatively small up to second order, increasing the order of approximation leads eventually to the divergence of the solution using the NP representation. As we discussed earlier, this cannot occur with the method proposed in this paper, with the LS method or with the NP method if "pruning" is applied, by construction.[25]

4 Conclusion

We have shown that the series-expansion method to solve non-linear equations discussed in the perturbation literature can be used to derive higher-order solutions for DSGE models that are recursively linear. This recursive linearity has the advantage of avoiding spurious diverging dynamics.

Our paper shows that perturbation methods need not be "problematic" as some recent literature has claimed. Obviously, there are limitation to the accuracy of the approximation that low-order perturbations can achieve for particular models. And, in some cases, the radius of convergence of the series expansion could be so small to make such an approach futile. Nevertheless, perturbation methods remain a very efficient way of analyzing a wide range of economic models, in some simple cases even analytically. Furthermore, very often perturbation methods remain the only viable solution when a large number of state variables is involved.

This paper sheds some light on the ambiguity existing in the current literature about the use of higher order state-space solutions. We show in particular that when

[25] We simulated the exact solution to the simple backward-looking equation $c_t = \gamma c_{t-1} + exp(\alpha c_{t-1}) + \sigma f_t$ where f_t is an i.i.d Gaussian process. We compared the simulation from the exact solution with that obtained with the approach proposed in this paper and with the non-pruned solution discussed in the literature. We considered up to six order of accuracy. The results confirmed that our solution can be arbitrarily close to the exact solution for parameter values that imply explosive solutions for the non-pruned approximation. The results are available from the author on request.

searching for a solution that is accurate to second order, the cross product terms must be computed using lower order terms. We show that this order criterion has formal foundations in the perturbation literature and the method of series expansions in particular.

By applying our solution method to the neo-classical growth model, we show that the solution method generates identical results to the method proposed by Lombardo and Sutherland (2007). Solutions that are not recursively linear, on the contrary, can generate diverging dynamics for large shocks and/or higher orders of approximation and, hence, depart dramatically from the solution proposed here. Stationarity is a very important property of our solution, as it allows to solve and simulate large DSGE models to higher order (e.g. for welfare analysis or in order to compute risk premia).

Our argument and exposition can be easily extended to any order of approximation.

References

Berglund, N. (2001). Perturbation theory of dynamical systems. *Arxiv preprint math.HO/0111178*.

Blanchard, O. J. and Kahn, C. M. (1980). The Solution of Difference Models under Rational Expectations. *Econometrica*, 48:1305–1311.

Bush, A. (1992). *Perturbation methods for engineers and scientists*. CRC.

Christiano, L. (2002). Solving dynamic equilibrium models by a method of undetermined coefficients. *Computational Economics*, 20(1):21–55.

Collard, F. and Juillard, M. (2001). Accuracy of stochastic perturbation methods: The case of asset pricing models. *Journal of Economic Dynamics and Control*, 25(6-7):979–999.

den Haan, W. and de Wind, J. (2010). How well-behaved are higher-order perturbation solutions? *DNB Working Papers*.

Gomme, P. and Klein, P. (2010). Second-order approximation of dynamic models without the use of tensors. *Journal of Economic Dynamics and Control*, Forthcoming.

Hinch, E. (1991). *Perturbation methods*. Cambridge Univ Pr.

Holmes, M. (1995). *Introduction to perturbation methods*. Springer.

Jin, H.-H. and Judd, K. J. (2002). Perturbation methods for general dynamic stochastic models. Manuscript, Stanford University.

Judd, K. L. (1998). *Numerical Methods in Economics*. MIT Press, Cambridge, MA.

Judd, K. L. (2002). Perturbation Methods with Nonlinear Changes of Variables. mimeo, Hoover Institution.

Judd, K. L. and Guu, S.-M. (2001). Asymptotic Methods for Asset Market Equilibrium Analysis. *Economic Theory*, 18:127–157.

Juillard, M. (1996). Dynare: A Program for the Resolution and Simulation of Dynamic Models with Forward Variables through the Use of a Relaxation Algorithm. CEPREMAP, Couverture Orange No. 9206.

Kelley, W. and Peterson, A. (2001). *Difference equations: an introduction with applications*. Academic Press.

Kim, J., Kim, S., Schaumburg, E., and Sims, C. A. (2008). Calculating and Using Second Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models. *Journal of Economic Dynamics and Control*, 32(11):3397–3414.

King, R. G. and Watson, M. W. (2002). System Reduction and Solution Algorithms for Singular Linear Difference Systems under Rational Expectations. *Computational Economics*, 20:57–86.

Klein, P. (2000). Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model. *Journal of Economic Dynamics and Control*, 24:1405–1423.

Ljungqvist, L. and Sargent, T. J. (2000). *Recursive Macroeconomic Theory*. MIT Press, Cambridge, MA.

Lombardo, G. and Sutherland, A. J. (2007). Computing Second-Order-Accurate Solutions for Rational Expectation Models using Linear Solution Methods. *Journal of Economic Dynamics and Control*, 31(2):515–530.

Schmitt-Grohé, S. and Uribe, M. (2004). Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. *Journal of Economic Dynamics and Control*, 28:755–775.

Swanson, E., Anderson, G., and Levin, A. (2006). Higher-order perturbation solutions to dynamic, discrete-time rational expectations models. Working Paper Series 2006-01, Federal Reserve Bank of San Francisco.

Uhlig, H. (1999). A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily. In Marimon, R. and Scott, A., editors, *Computational Methods for the Study of Dynamic Economies*, chapter 3. Oxford University Press, Oxford.

Table 1: Mean (x100) of simulation under three methods

Method[1]	Order	Capital	Consumption
Exact	–	0.9898	9.4294
TP (LS)	2	0.9231	8.7940
Dynare++(NP)[2]	2	0.9024	8.5966
Dynare++(NP)	3	1.0122	9.6425
Dynare++(NP)	4	-Inf	-Inf

[1] *All simulations are run with the same seed (10001) amending Dynare function "dynare_simul.m". We simulated 5000 periods with a standard deviation of 0.5 (shocks are standardized). To compute the mean the first 500 draws have been dropped.*

[2] *Using Dynare for the second order NP solution yields respectively 0.9015 and 8.5880. The small discrepancy with respect to Dynare++ is due to the different simulation algorithm used.*

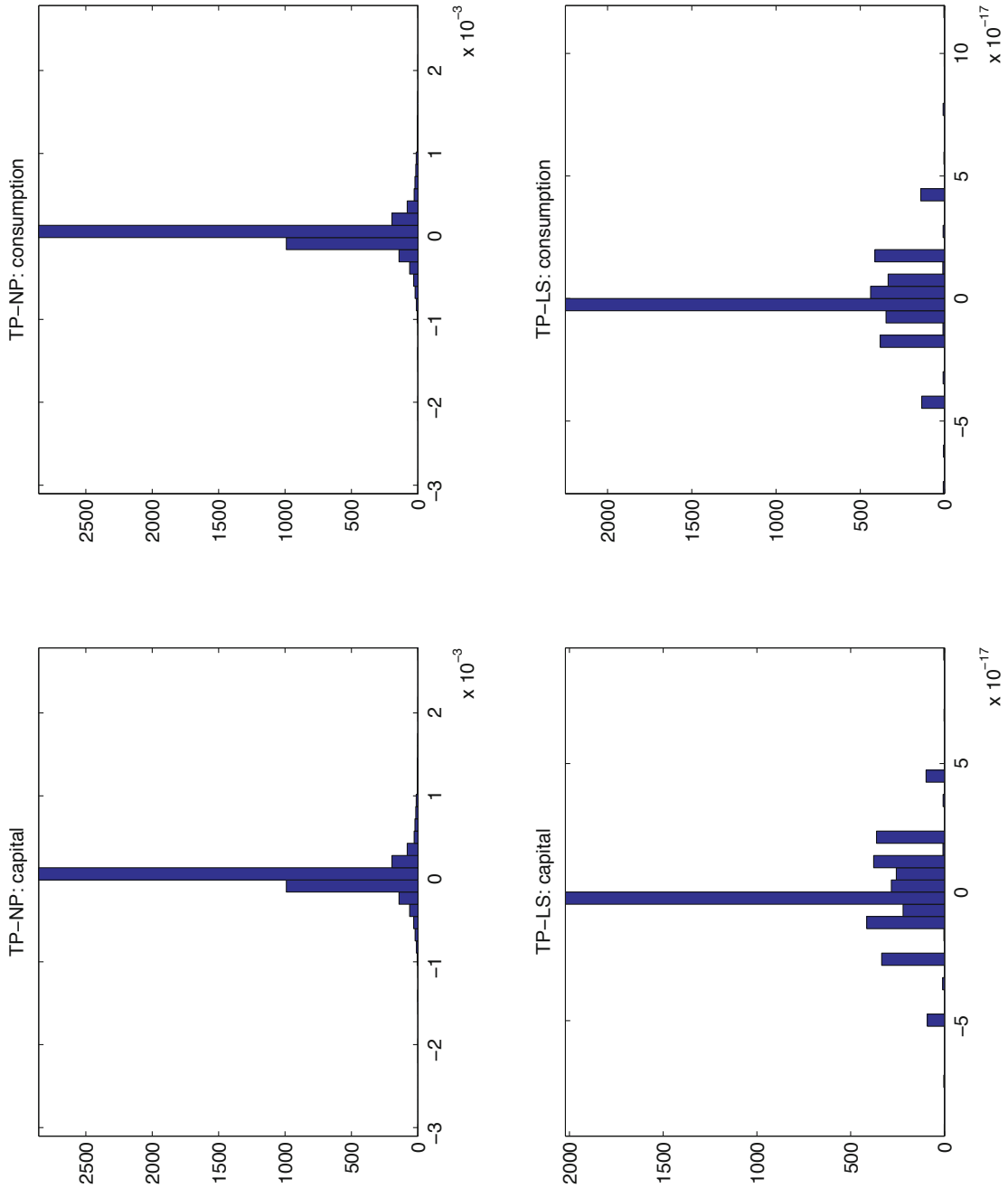

Figure 1: The case of small shocks: $\sigma = 0.1$

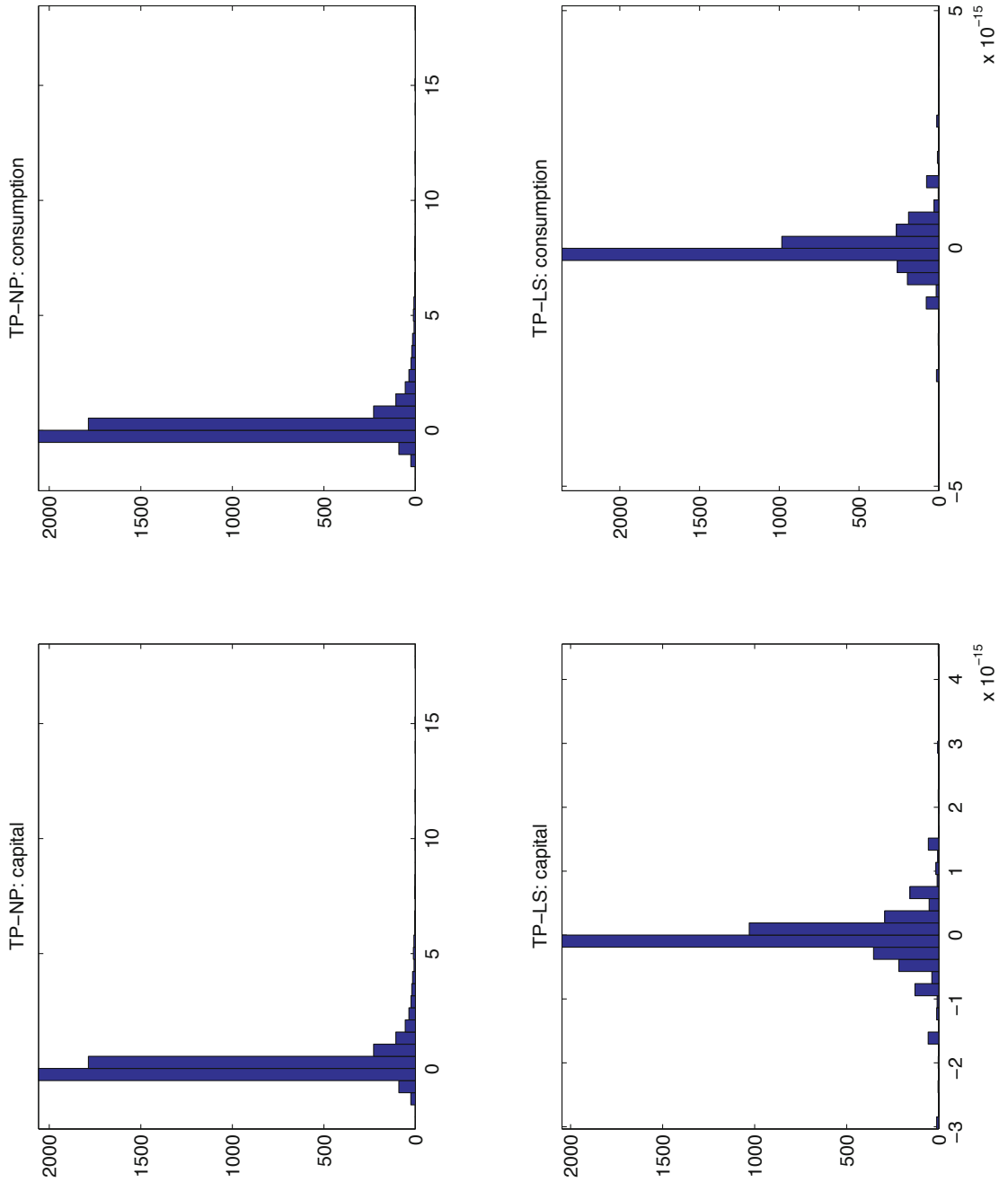

Figure 2: The case of large shocks: $\sigma = 1.5$