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2 The paper was produced whilst the author was at the European Central Bank for an internship in the Monetary Policy Strategy Division

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Abstract

This paper examines the feasibility of implementing Linear Quadratic Gaussian (LQG) Control in structural cointegrated VAR models and sheds some light on the two major problems generated by such implementation. The first aspect to be taken into account is the effect of the presence of unit roots in the system on the policymaker's ability to control it, partially or thoroughly. Different control techniques are proposed according to the extent to which the policymaker can exercise his control on the overall dynamics of the economy, i.e. depending on whether he/she can stabilize the whole system, only part of it or none of it. The second issue involves the structural form of the model. It will be shown in this paper that, in general, a system's features will change when implementing a new control rule. In particular, a controlled system will generally not retain features that should be intrinsecally invariant to policy changes (e.g., neutrality of money in the long-run).

Keywords: Optimal control, cointegration, policy invariance

JEL Classification: C32, C61, E52

Non Technical Summary

Structural VAR models are a predominant tool for monetary policy analysis. They appear to be able to identify the effects of policy without a complete structural model of the economy (as the Cowles Commission approach proposed), but using instead a very restricted number of identifying assumptions. In addition to this, these models have the very appealing characteristic of being able to incorporate the concept of cointegration and to, therefore, model in a sound manner the non-stationarity that characterizes most of the macroeconomic variables of interest for the policymaker (prices, money, etc..).

SVAR models give us the framework for the analysis of monetary policy strategies. Optimal control is then one of the main tools for modeling the central bank's decisional process and for, therefore, deriving "optimal" policy rules. The policymaker will define a loss function in terms of the deviations of the goal variables from their targets and will then minimize it under the constraints given by the dynamics of the economy, in order to obtain an optimal rule for the policy instruments. The literature regarding optimal control in VAR contexts applied to monetary policy is vast, but it has focused almost exclusively on implementing optimal control on stationary systems, which are easier to handle, but often less able to represent the data. In general, differentiation and filtering techniques are used to eliminate non-stationarity from the model with great loss of information. Our work focuses instead on incorporating optimal control in a non-stationary context, using in particular cointegrated VARs.

Following Monti and Mosconi (2003), we will show how it is possible to apply optimal control techniques to models that present non-stationarity and discuss the eventual issues and criticalities such implementation gives rise to. The feasibility of any control and the extent to which the policymaker can control the system depends on characteristics of the reduced form model like the cointegrating rank and the number of unit roots.

Moreover we will highlight a criticality that involves the structural form of the model. It will be shown that, in general, a system's structural features will change when implementing a new control rule. In particular, a controlled system will generally not retain features that should be intrinsically invariant to policy changes (e.g., neutrality of money in the long-run). The structural form of the model is, by definition, built to capture features of the model that are supposed to be invariant to changes in the policy regime. Our findings contradict this assumption. In order to control the system it is obvious that the control rule has to influence the dynamics of the non-policy variables: all coefficients of the system can be affected by the policy changes. This means that features of the system that are assumed to be invariant to policy are instead affected by policy changes.

We shall show this through two examples. The two models examined in this analysis are characterized by long-run neutrality of monetary policy on the real variables. This latter assumption is entrenched wisdom in modern macroeconomic theory and is generally built in as a feature in most macroeconomic models. In both models, nonetheless, the feature of long-run neutrality of money disappears. This paper acknowledges this shortcoming. Nevertheless, since the policy non-invariance property of certain features is essential for the soundness of the SVAR approach, future steps of research should seek to define conditions on the VAR coefficients that allow policy non-invariance.

1 Introduction

Structural Vector AutoRegressive models or SVARs, first conceived by Sims (1980,1986), have become a predominant tool for monetary policy analysis. The greatest appeal of using SVARs for evaluating monetary policy strategies and studying the monetary policy transmission mechanism is that they appear to be able to identify the effects of policy without a complete structural model of the economy (as the Cowles Commission approach proposed), but using instead a very restricted number of identifying assumptions. Bernanke and Mihov (1986), Christiano, Eichenbaum and Evans (1998) and Piffanelli (1999) are just a few examples of the application of the SVAR approach for monetary policy analysis.

SVAR models also have the very appealing characteristic of being able to incorporate the concept of cointegration and to, therefore, model in a sound manner the non-stationarity that characterizes most of the macroeconomic variables of interest for the policymaker (prices, money, etc..). In particular, the common-trends approach for the identification on SVARs (Warne, 1993) allows to model structural stochastic trends. We will focus exclusively on the latter identification strategy.

SVAR models give us the framework for the analysis of monetary policy strategies. Optimal control is then one of the main tools for modeling the central bank's decisional process and for therefore deriving "optimal" policy rules. The solution of an optimal control problem is, in fact, a motion-law for the policy instrument, that minimizes the central bank's loss function under the constraints given by the dynamics of the economy. In general the central bank's loss function is modeled so to be quadratic and to assign a cost to deviations of some objective-variables from their targets and, possibly, to the differences of the policy instruments (interest rate smoothing). Other functional forms have been proposed to model the central bank's loss function (e.g., the asymmetric linex function proposed by Ruge'-Murcia, 2001), in order to achieve a higher degree of realism in its description. The great majority of the literature has, nonetheless, continued to focus on quadratic loss functions and we will do so too.

The literature regarding the definition of optimal control rules using optimal control applied in VAR contexts is vast: a very incomplete list of relevant articles would include Ball (1999), Taylor (1999) and Rudebusch and Svensson (2001, 1999). This massive literature, however, focuses mainly on implementing optimal control on stationary systems. One of the few attempts to tackle the issue of controlling a cointegrated VAR can be found in the Johansen and Juselius (2001) paper "Controlling Inflation in a Cointegrated Vector AutoRegressive Model with an Application to U.S. data". Johansen and Juselius develop a control technique for reduced form cointegrated VARs that returns a so-called *instrument rule*[1], i.e. a rule that does not derive from any type of optimization problem. Given the fact that cointegrated SVARs seem able to describe more closely the economy's functioning and that the policymaker's decisional process

[1] Refer to Rudebusch e Svennson (1999) for a definition of instrument rules and targeting rules.

seems to be reasonably well approximated by a linear quadratic (LQ) optimal control problem, our goal is to define *targeting rules* (i.e. rules that derive from an optimization problem, defined as in Rudebusch e Svennson (1999)) and understand the issues related to implementing optimal control on cointegrated SVAR models and the criticalities such implementation eventually gives rise to.

Such issues and criticalities are numerous and involve different aspects of the problem: on the one hand, there are issues more related with the reduced form model (i.e. is the system stabilizable?, how many unit roots can be removed?, etc..), on the other hand, there are the criticalities which are more related to the properties of the structural model (i.e., are the structural features identified before control maintained after control? If not, what conditions are to be met for such features to hold before and after control?). We will, therefore, organize our analysis in the following way. Section 2 focuses on how to cast a cointegrated structural VAR into a representation that makes it suitable for the application of optimal control techniques. In particular, subsection 2.1 briefly describes cointegrated VARs and the common-trends approach, subsection 2.2 illustrates the optimal control problem and subsection 2.3 shows how to define the VAR so to be compatible with the standard control problem. Section 3 focuses on how the presence of unit roots affects the feasibility of control and the ways of controlling a system. Section 4 discusses how the implementation of an optimal control rule derived on the basis of this may transform the structure of the system. In section 5 we present a number of empirical examples to illustrate the validity of the findings of the previous sections. Section 6 concludes.

2 Bridging Optimal Control and Cointegrated Structural VARs

This section mainly summarizes the most important results of Vector AutoRegressions theory and of control theory and demonstrate how to express VARs so to be in line with the Control theory representation of the problem.

2.1 Cointegrated VARs and the Common Trends approach to Structuralisation

Consider the p-dimensional VAR(k), cointegrated I(1) with rank r:

$$z_t = \sum_{i=1}^{k} \Pi_i z_{t-i} + \Phi D_t + \varepsilon_t \tag{1}$$

where D_t represents the deterministic term and can contain a constant, a linear term, seasonal dummies, intervention dummies, or any other regressor that we consider fixed and non stochastic. The residuals ε_t are independent and identically distributed as a gaussian with mean zero and variance matrix Ω, i.e. $\varepsilon_t \sim i.i.d.N(0, \Omega)$. The model can be expressed in the vector error correction form

$$\Delta z_t = \alpha\beta' z_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta z_{t-i} + \Phi D_t + \varepsilon_t$$

where α and β are $(p \times r)$-dimensional matrices and represent, respectively, the loadings matrix and the cointegration matrix. Based on Johansen (1996) we obtain the Vector Moving Average representation (VMA, from now on)

$$z_t = A_t + C \sum_{i=0}^{t-1} (\varepsilon_{t-i} + \Phi D_{t-i}) + C^*(L)(\varepsilon_t + \Phi D_t) \tag{2}$$

where A_t is dependent on initial values and is such that $\beta' A = 0$, with $A = \lim_{t->\infty} A_t$, $C^*(L) = I + \sum_{i=0}^{\infty} C_i^*$ and $C = \beta_\perp(\alpha'_\perp \Gamma \beta_\perp)^{-1}\alpha'_\perp$, with $\Gamma = I - \sum_{i=1}^{k-1} \Gamma_i$. It can be seen from equation 2 that the non-stationarity in the process is generated by the cumulated sum of the residuals, or, more precisely, by the linear combinations $\alpha'_\perp \sum_{i=0}^{t-1} \varepsilon_{t-i}$, the so-called common trends of the model. Since the model is cointegrated I(1), the number of unit roots in the model is exactly $s = p - r$.[2]

Let us assume, for the sake of notational simplicity, but without loss of generality, that the deterministic component of the model is given by a constant $\mu_0 \nsubseteq Sp(\alpha)$ that it gives rise to a linear trend and a linear term $\mu_1 t$, with $\mu_1 \subseteq Sp(\alpha)$, so that it does not generate a quadratic trend. The VMA form of the model will then be

$$z_t = C \sum_{i=0}^{t-1} \varepsilon_{t-i} + C^*(L)\varepsilon_t + \tau_0 + \tau_1 t$$

where the functional forms of τ_0 and τ_1 are derived in appendix A.

The relationship between the reduced form residuals $\varepsilon_t \sim i.i.d.N(0, \Omega)$ and the structural form innovations $e_t \sim i.i.d.N(0, I)$ is assumed to be

$$\varepsilon_t = B e_t, \tag{3}$$

where B is a $p \times p$ non singular matrix. It is therefore possible to write

$$z_t = CB \sum_{i=0}^{t-1} e_{t-i} + C^*(L)Be_t + \tau_0 + \tau_1 t. \tag{4}$$

The CB matrix contains the long run impact coefficients for the structural innovations. The reduced form parameters can be estimated with maximum likelihood techniques (Johansen, 1996). The identification of B is obtained in the following manner. First of all the hypothesis of orthonormality of the structural innovations, i.e.,

[2] We will retain the standard assumption of VAR theory that all the roots of the characteristic polynomial of the VAR, $A(L) = I - \sum_{i=1}^{k} \Pi_i L^i$, are all lying outside or on the unit circle. The literature that discards this assumption is very small. See Juselius and Mladenovic (2001) for an example.

$$E(e_t e_t') = B^{-1}\Omega B^{-1\prime} = I_p$$

places $p(p+1)/2$ non-linear restrictions on B. To obtain just-identification, other $p(p-1)/2$ restrictions are needed. Following Warne (1993) $(p-r)r$ restrictions are found by assuming that only the first $s = p - r$ structural innovations, φ_t, are permanent (cumulated in X_t), while the rest, ν_t, are transitory. This implies that the following restrictions have to be imposed on the long run impact matrix CB

$$CBU = 0_{p,r}$$

where $U = \begin{bmatrix} 0_{p-r,r} & I_r \end{bmatrix}'$ extracts the last r rows of CB. $e_t = \begin{bmatrix} \varphi_t' & v_t' \end{bmatrix}'$. Since $C = \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp'$, one can write

$$\alpha_\perp' BU = 0_{p-r,r}. \tag{5}$$

The other restrictions needed can be derived from hypotheses regarding the long run effect of permanent innovations and the instantaneous impact of both transitory and permanent innovations. Economic models used to describe the economy for monetary policy analysis generally comprise real variables (different measures of output, real money balances, etc...), nominal variables to be controlled (inflation indices, long-term interest rate, etc.) and a control instrument, typically a short term interest rate. It is common practice (Blanchard and Quah, 1989 and Warne, 1993 are just the most famous examples) to impose constraints on the long-run impact matrix that compel the nominal variables (in particular inflation) not to have long-run effects on the real variables (i.e., the output). This feature is considered to be policy invariant.

2.2 The Standard Linear Quadratic Gaussian Control Problem

Consider the following first order linear system:

$$X_t = AX_{t-1} + Bu_{t-1} + \varepsilon_t \tag{6}$$

$$Y_t = HX_t + Ju_t$$

where X_t and u_t are N-dimensional and m-dimensional and represent, respectively, the state of the system at time t and the vector of control variables at time t. Y_t is a μ-dimensional vector of target-variables and ε_t is an N-dimensional vector of disturbances, white noise with mean zero and covariance matrix given by $E_t(\varepsilon_{t+1}\varepsilon_{t+1}') = \Omega$. The goal of an optimal control problem is to find, if it exists, a sequence of controls

$$u_{[t_0,T)} = \{u_t\}_{t=t_0}^{\infty}$$

during the whole regulation period $[t_0, \infty)$ (which we assume to be infinite, in line with the greater part of economic literature), that minimizes the loss function defined as:

$$L = \sum_{t=t_0}^{\infty} Y_t' K Y_t$$

or

$$L = \sum_{t=t_0}^{\infty} \begin{bmatrix} X_t' & u_t' \end{bmatrix} \begin{bmatrix} Q & W \\ W' & R \end{bmatrix} \begin{bmatrix} X_t \\ u_t \end{bmatrix}$$

where $Q = H'KH, W = H'KJ$ and $R = J'KJ$ are assumed to be symmetric matrices. In the standard LQG control problem Q and W are assumed to be positive semidefinite and R positive definite[3]. The control problem can then be written in the following way:

$$\min_{u_t} \sum_{t=t_0}^{\infty} \begin{bmatrix} X_t' & u_t' \end{bmatrix} \begin{bmatrix} Q & W \\ W' & R \end{bmatrix} \begin{bmatrix} X_t \\ u_t \end{bmatrix} \tag{7}$$

$$s.t. \quad X_t = AX_{t-1} + Bu_{t-1} + \varepsilon_t$$

with X_0 given. The two latter formulations of the control problem are absolutely equivalent, so, from now on, we shall refer solely to model (7), bearing in mind that Q, W and R are obtained from the costs on the goal variables as described just above.

The control problem (7) can be solved, if a solution exists, using dynamic programming techniques (refer to Mosca, 1995, Bertsekas, 1995 or Anderson and Moore, 1989 for a detailed analyses of optimal control and dynamic programming). The existence of a solution is conditional to the fulfillment of certain regularity conditions we will describe just below. Before illustrating the solution of the control problem (7), one important remark has to be made. When the vector of disturbances that affects the system is white noise with mean zero and serially non correlated, the so-called certainty equivalence principle holds and states that the optimal control rule for the stochastic system is equivalent to the optimal rule for the deterministic system, except for the fact that it responds to efficient estimates of the state-variables vector, instead of their actual value. Since the assumption of "white-noiseness" underlies all our work, we simply derive the deterministic solution.

Now, consider again problem (7), assuming that $\varepsilon_t = 0$ for $t = t_0, t_0 + 1, t_0 + 2, ...$ It is possible to demonstrate that, if the system (6) is stabilizable and detectable[4], the solution of control problem (7) is

$$u_t = -FX_t \tag{8}$$

[3] If the latter assumption is removed, i.e. if the instrument is allowed to be costless, i.e. $R = 0$, the control problem takes the name of cheap control problem

[4] A system $\Sigma = (A, B, H, J)$ is said to be stabilizable if there exist matrices F $(m \times n)$ such that the matrix $A - BF$ has all eigenvalues strictly contained in the unit circle. A system $\Sigma = (A, B, H, J)$ is said to be detectable if there exist matrices K $(n \times \mu)$ such that the matrix $A + KH$ has all eigenvalues strictly contained in the unit circle.

where

$$F = (R + B'\widehat{P}B)^{-1}(B'\widehat{P}A + W')$$

and $\widehat{P}$ is the solution of the so called Algebraic Riccati Equation (ARE),

$$\widehat{P} = Q + A'\widehat{P}A - (A'\widehat{P}B + W)(R + B'\widehat{P}B)^{-1}(B'\widehat{P}A + W').$$

A detailed analysis of computational techniques and algorithms for solving the ARE can be found in Anderson, Hansen, McGrattan and Sargent (1996).

The possibility of driving target variables on desired means or trends will be analyzed in section 3. Anyhow, it can be proven that the solution of such control problem is the following policy rule:

$$u_t = -FX_t + \nu(t),$$

where F is the solution of the same control problem without deterministic trends and the derivation of $\nu(t)$ will be shown soon. Refer to Monti and Mosconi (2003) for the demonstration of why the two parts of the control rule can be defined separately.

2.3 Bridging the two theories

The VAR(k) model displayed in equation (1) is a closed-loop model (i.e., containing the equation for the control instrument) with k lags and deterministic components, while the system in equation (6) is a first order homogeneous system in which the control instrument is considered exogenous. The model in equation (1) can, however be expressed as the model in equation (6).

Consider again the p-dimensional VAR(k), cointegrated I(1) with rank r ($s = p - r$ unit roots):

$$z_t = \sum_{i=1}^{k} \Pi_i z_{t-i} + \mu_0 + \mu_1 t + \varepsilon_t$$

where $\mu_0 \nsubseteq Sp(\alpha)$ and $\mu_1 \subseteq Sp(\alpha)$ and the residuals ε_t are independent and identically distributed as a Gaussian with mean zero and variance matrix Ω, i.e. $\varepsilon_t \sim i.i.d.N(0, \Omega)$. The solution to this model is

$$z_t = C\sum_{i=0}^{t-1} \varepsilon_{t-i} + C^*(L)\varepsilon_t + \tau_0 + \tau_1 t$$

where the functional forms of τ_0 and τ_1 are derived in appendix A. The first step is to transform the VAR model with deterministic components into a homogenous VAR. This can be easily accomplished by defining a new variable $\overline{z}_t = z_t - \tau_0 - \tau_1 t$: this new process $\overline{z}_t$ is represented by a VAR cointegrated I(1) analogous to the previous one, but devoid of deterministic components, i.e.

$$\overline{z}_t = \sum_{i=1}^{k} \Pi_i \overline{z}_{t-i} + \varepsilon_t. \tag{9}$$

Let us now assume that $\overline{z}_t$ can be divided in two subvectors x_t $(n\times 1)$ and u_t $(m \times 1)$ with $m = p - n$, respectively containing the non-policy macroeconomic variables and policy (control) variables. Model (9) can then be rewritten as:

$$\begin{bmatrix} x_t \\ u_t \end{bmatrix} = \sum_{i=1}^{k} \begin{bmatrix} \Pi_{i.11} & \Pi_{i.12} \\ \Pi_{i.21} & \Pi_{i.22} \end{bmatrix} \begin{bmatrix} x_{t-i} \\ u_{t-i} \end{bmatrix} + \varepsilon_t$$

This model can be straightforwardly expressed as a VAR(1), using the so-called "companion form" transformation, i.e.

$$\widetilde{Z_t} = \widetilde{\Pi}_1 \widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t \ \ or \ \ \Delta\widetilde{Z_t} = \widetilde{\alpha}\widetilde{\beta}' \widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t \tag{10}$$

where

$$\widetilde{Z_t} = \begin{bmatrix} \overline{z}_t \\ \overline{z}_{t-1} \\ \vdots \\ \overline{z}_{t-k+1} \end{bmatrix}, \ \widetilde{\Pi}_1 = \begin{bmatrix} \Pi_1 & \Pi_2 & & \Pi_k \\ I_p & 0 & ... & 0 \\ \vdots & & \ddots & \vdots \\ 0 & ... & I_p & 0 \end{bmatrix}, \ \widetilde{\varepsilon_t} = \begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$\widetilde{Z_t}$ is $(p \times k)$-dimensional. This model has the same number of unit roots as model (9), i.e. s, and will therefore have rank $\widetilde{r} = pk - s$. $\widetilde{\alpha}$ and $\widetilde{\beta}$ are $(pk \times \widetilde{r})$ matrices, defined in the following way:

$$\widetilde{\alpha} = \begin{bmatrix} \alpha & \Gamma_1 & ... & \Gamma_{k-1} \\ 0 & I_p & & 0 \\ \vdots & & \ddots & \\ 0 & ... & 0 & I_p \end{bmatrix}, \ \widetilde{\beta} = \begin{bmatrix} \beta & I_p & 0 & 0 \\ 0 & -I_p & & 0 \\ \vdots & & \ddots & I_p \\ 0 & ... & 0 & -I_p \end{bmatrix}$$

As mentioned above, we shall use the common-trends approach to identify the system. In such case, the relationship between the reduced form residuals $\widetilde{\varepsilon}_t \sim i.i.d.N(0, \Omega)$ and the structural form innovations $\widetilde{e}_t \sim i.i.d.N(0, I)$ is assumed to be:

$$\widetilde{\varepsilon}_t = \widetilde{B}^{str}\widetilde{e}_t,$$

where B is a $p \times p$ non singular matrix. The identification strategy has been described above, in subsection 2.1. The structural model will be

$$\widetilde{Z_t} = \widetilde{\Pi}_1 \widetilde{Z}_{t-1} + \widetilde{B}^{str}\widetilde{e_t} \ \ or \ \ \Delta\widetilde{Z_t} = \widetilde{\alpha}\widetilde{\beta}' \widetilde{Z}_{t-1} + \widetilde{B}^{str}\widetilde{e_t}. \tag{11}$$

Comparing equations (10) and (11), it is easy to see that the coefficients for the lags are the same. It is very useful to underline this feature, since it will allow us to plainly extend the analysis of controllability done on the reduced form to the structural form.

Model (11) is a homogeneous first order linear system, but it is still not in the form of model (6), because the policy rule is still endogenous, that is the model still contains the "old" control rule for the policy instrument. Model (11) is, so to say, the closed loop system obtained by implementing the "historical" estimated policy rule. Our goal is to redefine the system so to be able to solve an optimal control problem, i.e., to find a new "optimal" motion-law for the policy instruments u_t. That means we will have to consider the open-loop system, i.e. the model obtained by removing the equations for the policy instruments and by considering them exogenous entries. We have assumed that $\overline{z}_t$ can be divided in two subvectors x_t $(n\times 1)$ and u_t $(m\times 1)$ with $m=p-n$, respectively containing the non-policy macroeconomic variables and policy variables. The first order linear system we will actually use as constraint for our optimization will then be:

$$X_t = AX_{t-1} + Bu_{t-1} + \widehat{B}^{str}\hat{e}_t \ \ or \ \ \Delta X_t = \widehat{\alpha}\widehat{\beta}' X_{t-1} + Bu_{t-1} + \widehat{B}^{str}\hat{e}_t \quad (12)$$

where $X_t = \begin{bmatrix} x_t & x_{t-1} & ... & x_{t-k+1} & u_{t-1} & ... & u_{t-k+1} \end{bmatrix}'$ is an N-dimensional matrix, with $N = nk + m(k-1)$[5],and

$$A = \begin{bmatrix} \Pi_{1.11} & ... & \Pi_{k.11} & \Pi_{1.12} & ... & \Pi_{k.12} \\ I_n & 0_{n,n} & ... & & & 0_{n,m} \\ 0_{n,n} & \ddots & & & & \vdots \\ \vdots & & I_n & & & \\ 0_{m,n} & ... & 0_{m,n} & 0_{m,m} & ... & 0_{m,m} \\ & & & I_m & & \vdots \\ \vdots & & & & \ddots & 0_{m,m} \\ 0_{m,n} & ... & & ... & 0_{m,m} & I_m \end{bmatrix}, \ B = \begin{bmatrix} \Pi_{1.12} \\ 0 \\ \vdots \\ 0 \\ I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\widehat{\alpha} = \begin{bmatrix} \alpha_{n,r} & \Gamma_{1.11} & ... & \Gamma_{k-1.11} & \Gamma_{1.12} & ... & \Gamma_{k-1.12} \\ 0_{n,r} & I_n & 0_{n,n} & & ... & & 0_{n,m} \\ \vdots & & \ddots & & & & \\ 0_{n,r} & & & I_n & 0_{n,m} & & \vdots \\ 0_{m,r} & & & 0_{m,n} & I_m & & \\ \vdots & & & & & \ddots & 0_{m,m} \\ 0_{m,r} & ... & & ... & & 0_{m,m} & I_m \end{bmatrix}$$

and

$$\widehat{\beta} = \begin{bmatrix} \beta_{n,r} & I_n & 0_{n,n} & ... & 0_{n,m} \\ 0_{n,r} & -I_n & I_n & & \vdots \\ \vdots & & \ddots & & 0_{m,m} \\ & & & \ddots & I_m \\ 0_{m,r} & .. & .. & 0_{m,n} & -I_m \end{bmatrix}.$$

[5] Remember that we have assumed that $\overline{z}_t$ can be divided in two subvectors x_t $(n\times 1)$ and u_t $(m\times 1)$ with $m=p-n$, respectively containing the non-policy macroeconomic variables and policy (control) variables

Now, system (12) is in the form of model (6) and can be used as a constraint for the optimization. We have extracted an equation from the system, rendering the control variables exogenous. This can produce deep changes in the features of the model. There, in fact, is no guarantee that model (12) will maintain the same number of unit roots as model (6). In addition, the eigenvalues of the model will change, and nothing assures that they will lay inside or on the unit circle. These aspects are relevant in determining whether the system can be controlled, partially or completely, so we will analyze these issues in the following section.

3 Controlling Cointegrated VARs: consequences of the presence of unit roots on the feasibility of control

The main focus of this section is to define the conditions under which a system like (12), or some of its variables, can be made stationary around a given mean or trend. This depends on whether the rank of the open-loop system can be augmented and to what extent. Notice that in the following section we will consider the system as if it were deterministic, because the residuals are white-noise with mean zero and serially non-correlated: this ensures that the certainty equivalence principle (mentioned in subsection 2.2) holds.

Monti and Mosconi (2003) have derived the conditions under which it is possible to remove unit roots from the system and we will examine them in depth. Consider again the VAR(1) presented in equation (12) in its VECM form, i.e.:

$$\Delta X_t = \widehat{\alpha}\widehat{\beta}' X_{t-1} + Bu_{t-1}.$$

Let us assume that the rank of this system is r_{ol}, i.e. $rank(\widehat{\alpha}\widehat{\beta}') = r_{ol}$, and that $B = a_u b_u'$, where a_u and b_u are full rank matrices. Any policy rule of the form

$$u_t = -FX_t,$$

will lead to the closed loop vector-equation for the non-policy variables

$$\Delta X_t = (\widehat{\alpha}\widehat{\beta}' - BF)X_{t-1}$$

or

$$\Delta X_t = \begin{bmatrix} \widehat{\alpha} & a_u \end{bmatrix} \begin{bmatrix} \widehat{\beta}' \\ -b_u' F \end{bmatrix} X_{t-1}. \tag{13}$$

Let us now define $r_{\max} = rank \begin{bmatrix} \widehat{\alpha} & a_u \end{bmatrix} \leq r$. Since the intersection of $Sp(\widehat{\alpha})$ and $Sp(a_u)$ might not be empty, the following inequalities will hold:

$$r_{ol} \leq r_{\max} \leq r_{ol} + m \tag{14}$$

These inequalities have several implications:

- The number of stationary linear combinations in the controlled closed-loop system, say r_{cl}, depends on F and is upward bounded by

$$r_{cl} \leq r_{\max}.$$

  Therefore, in order to make X_t stationary, i.e. $r_{cl} = n$, it is necessary that $r_{\max} = n$. That is equivalent to saying that a_u has $n - r_{ol}$ columns that are linearly independent on $\widehat{\alpha}$. This case is examined in section 3.1.

- When the policy variable is switched off, i.e. $u_t = 0$, the number of unit roots in X_t is $n - r_{ol}$. Therefore, an active policy can remove at most $r_{\max} - r_{ol}$ unit roots. The maximum number of unit roots that can be removed from the system (13) is m, i.e. it coincides with the number of instruments. The case in which some unit roots can be removed, but the controlled system will still maintain some degree of non-stationarity is treated in section 3.2

- When $r_{\max} = r_{ol}$, that is when $B \subseteq Sp(\widehat{\alpha})$, it is impossible to reduce the number of unit roots in the feedbacked system (13). In this case it is still possible to transform up to m of the existing cointegrating vectors into others, that are more convenient for the policymaker. Apart from very specific cases, the instruments vector u_t will have to be non-stationary, to be able to modify the cointegration vectors. Section 3.3 examines this case, i.e. when $B \subseteq Sp(\widehat{\alpha})$, with greater detail.

3.1 Case 1: the system is stabilizable

If the system is stabilizable[6] in the control theoretic sense, it means that $r_{\max} = n$ and that the standard LQ control problem can be solved on such system and a control rule of the type in equation (8) will be obtained, i.e.

$$u_t = -FX_t$$

where

$$F = (R + B'\widehat{P}B)^{-1}(B'\widehat{P}A + W')$$

and $\widehat{P}$ is the solution of the so called Algebraic Riccati Equation (ARE),

$$\widehat{P} = Q + A'\widehat{P}A - (A'\widehat{P}B + W)(R + B'\widehat{P}B)^{-1}(B'\widehat{P}A + W').$$

As mentioned above, in Monti and Mosconi (2003) it has been demonstrated that, if one wants to drive the target variables on a specific target values or trends, a rule of this form, $u_t = -F\widehat{X}_t + \nu_0 + \nu_1 t$, is needed and that it is possible to derive the two parts of the rule separately. The derivation of F has been briefly treated in 2.2: we shall now, following Monti and Mosconi (2003), describe how to derive ν_0 and ν_1.

[6] We assume the system is detectable.

Let us turn again to the initial control problem:

$$\min_{u_t} \sum_{i=0}^{\infty} \left[\begin{array}{cc} X'_{t+i} & u'_{t+i} \end{array} \right] \left[\begin{array}{cc} Q & W \\ W' & R \end{array} \right] \left[\begin{array}{c} X_{t+i} \\ u_{t+i} \end{array} \right]$$

$$s.t. \qquad X_t = AX_{t-1} + Bu_{t-1} + \mu_0 + \mu_1 t$$

where $(A - I) = \widehat{\alpha}\widehat{\beta}'$ and $\mu_0 \nsubseteq Sp(\widehat{\alpha})$ and $\mu_1 \subseteq Sp(\widehat{\alpha})$. Recall that $X_t = \left[\begin{array}{cccccc} x_t & x_{t-1} & ... & x_{t-k+1} & u_{t-1} & ... & u_{t-k+1} \end{array} \right]'$ is an N-dimensional matrix, with $N = nk + m(k-1)$ and u_t is m-dimensional.

Consider then the policy rule

$$u_t = -FX_t + \nu_0 + \nu_1 t.$$

All the eigenvalues of $(A - BF')$ are strictly inside the unit circle. It is straightforward to show that, when such policy is adopted, the system will converge to

$$\lim_{t \longrightarrow \infty} \left\{ X_t + (\alpha\beta' - BF)^{-1} B(\nu_0 - \nu_1 + \nu_1 t) \right\}, \tag{15}$$

irrespective of the initial condition. At first we shall derive the conditions under which it is possible to define v such that $X_\infty = \xi_0 + \xi_1 t$. Finally we will show how only some variables of vector X_t can be targeted. But first we will prove the following theorem (derived in Monti and Mosconi, 2003)

Theorem 1 *A sufficient condition to make X_t stable and asymptotically equal to $\xi_0 + \xi_1 t$, is that $m \geq n$. However, this condition is not necessary: the necessary and sufficient condition is that*

$$\xi_0 \subseteq Sp\,(R_\perp)$$

$$\xi_1 \subseteq Sp\,(R_\perp)$$

where $R = \widehat{\beta}\widehat{\alpha}'\overline{B}_\perp$, with $\overline{B}_\perp = B_\perp (B'_\perp B_\perp)^{-1}$.
Proof. *Letting $X_\infty = \xi$ and recalling that $B_\perp \overline{B}'_\perp + B\overline{B}' = I$ (with $\overline{B}$ defined analogously to $\overline{B}_\perp$), equation (15) can be rewritten as*

$$\xi = -\left(\left[\begin{array}{cc} B_\perp & B \end{array} \right] \left[\begin{array}{c} \overline{B}'_\perp \alpha\beta' \\ \overline{B}'\alpha\beta' - F \end{array} \right] \right)^{-1} B(\nu_0 - \nu_1 + \nu_1 t).$$

Let us now define $R = \widehat{\beta}\widehat{\alpha}'\overline{B}_\perp$ and $G = \widehat{\beta}\widehat{\alpha}'\overline{B} - F'$. Developing the inverse we obtain

$$\xi = -\left[\begin{array}{cc} G_\perp (R' G_\perp)^{-1} & R_\perp (G' R_\perp)^{-1} \end{array} \right] \left[\begin{array}{c} \overline{B}'_\perp \\ \overline{B}' \end{array} \right] B\nu$$

or

$$\xi_0 = -R_\perp (G' R_\perp)^{-1} (\nu_0 - \nu_1) \tag{16}$$

$$\xi_1 = -R_\perp (G' R_\perp)_1^{-1} \nu_1$$

This equation may be solved for F and v if and only if $\xi_0 \subseteq Sp\,(R_\perp)$ and $\xi_1 \subseteq Sp\,(R_\perp)$. ■

It is very important to highlight the following aspect of this theorem. Since $Sp(R_\perp)$ depends solely on A and B, it is possible to solve the control problem, first determining F as if $X_\infty = 0$, i.e. solving a standard LQG regulator problem, and then determine ν. Once F is determined the solution for ν is:

$$\nu_1 = -G' R_\perp \overline{R}'_\perp \xi_1$$

$$\nu_0 = -G' R_\perp \overline{R}'_\perp (\xi_0 - \xi_1)$$

The policymaker generally cannot target all the variables in the system and does not have as many instruments as targets, so it is useful to analyze the case in which $m < n$ and not all the variables are targeted by the policymaker. More specifically, assume that the target vector is $S'X_t = s_0 + s_1 t$, where S is a $n \times m$ full rank matrix and s_0 and s_1 are an m-dimensional vector. This target may be easily achieved by finding the optimal F and then solving

$$S'X_\infty = -S'R_\perp (G'R_\perp)^{-1} (\nu_0 - \nu_1 + \nu_1 t) = s_0 + s_1 t$$

for v, obtaining

$$\begin{aligned} \nu_1 &= -(G'R_\perp)(S'R_\perp)^{-1} s_1. \\ \nu_0 &= -(G'R_\perp)(S'R_\perp)^{-1} (s_0 - s_1). \end{aligned} \tag{17}$$

This technique for deriving the deterministic part of the control rule, i.e. the coefficients ν_0 and ν_1, will also be used in the next section.

3.2 Case 2: the system is partially stabilizable

In this case, i.e. when $r_{\max} \geq r_{cl}$, the number of unit roots present in the system can be reduced, but the open-loop system cannot be made completely stationary. The optimal control rule can be found with the following procedure. Consider the following system

$$X_t = AX_{t-1} + Bu_{t-1} + \mu_0 + \mu_1 t \;\; or \;\; \Delta X_t = \widehat{\alpha}\widehat{\beta}' X_{t-1} + Bu_{t-1} + \mu_0 + \mu_1 t \tag{18}$$

where $X_t = \begin{bmatrix} x_t & x_{t-1} & \ldots & x_{t-k+1} & u_{t-1} & \ldots & u_{t-k+1} \end{bmatrix}'$.

Assume that, $r_{\max} = r_{ol} + m$, i.e. m unit roots can be removed from the non-policy variable equations. Using the following transformation, it is possible to clearly separate the stabilizable part of system (18) from the non-stabilizable part, $\overline{X}_t = TX_t$

$$\overline{X}_t = \begin{bmatrix} u_{t-1} \\ \Delta x_t \\ \vdots \\ \Delta x_{t-k+2} \\ \Delta u_{t-1} \\ \vdots \\ \Delta u_{t-k+2} \\ \beta' X_{t-k+1} \\ c' X_t \\ d' X_t \end{bmatrix} T = \begin{bmatrix} 0_{m,n} & ... & 0_{m,n} & I_m & ... & 0_{m,m} \\ I_n & -I_n & & & & \\ 0_{n,n} & \ddots & 0_{n,n} & ... & ... & 0_{n,m} \\ \vdots & I_n & -I_n & 0_{n,m} & & \\ & 0_{m,n} & & I_m & -I_m & \vdots \\ \vdots & & & & \vdots & 0_{m,m} \\ & & & 0_{m,m} & I_m & -I_m \\ 0_{r,m} & ... & \widehat{\beta}' & 0_{r,m} & ... & 0_{r,m} \\ c' & 0_{m,n} & & ... & & 0_{m,m} \\ d' & 0_{s-m,n} & & ... & & 0_{s-m,n} \end{bmatrix}$$

where T is an invertible $N-$dimensional matrix, c is a m-dimensional matrix that extracts the m variables that the policymaker chooses to control with the m instruments to him available and d has dimensions $(s-m) \times n$ and extracts the linear combinations that the policymaker has chosen not to stabilize. The first $N-s$ rows of $\overline{X}_t$ are stationary by definition: the model is I(1), so the lagged differences of X_t and u_t are stationary, as are the vectors containing the cointegrating relationships $\beta' X_{t-k+1}$. The transformed system will then be:

$$\overline{X}_t = \overline{AX}_{t-1} + \overline{B}u_{t-1} + \overline{\mu}_0 + \overline{\mu}_1 t$$

where $\overline{A} = TAT^{-1}$, $\overline{B} = TB$, $\overline{\mu}_0 = T\mu_0$ and $\overline{\mu}_1 = T\mu_1$ and $\overline{A}$ and $\overline{B}$ have the following structure:

$$\overline{A} = \begin{bmatrix} M_{N-s} & 0_{N-s,s} \\ V_{s,N-s} & I_s \end{bmatrix} \quad and \quad \overline{B} = \begin{bmatrix} \overline{B}_R \\ \overline{B}_{nonR} \end{bmatrix}$$

with M

$$M = \begin{bmatrix} 0_{m,m} & 0_{m,n} & & ... & & ... & & 0_{m,r} \\ 0_{n,m} & \Gamma_{1.11} & ... & \Gamma_{k-2.11} & \Gamma_{1.12} & ... & \Gamma_{k-1.12} & \widehat{\alpha} \\ \vdots & I_n & & & & & & 0_{n,r} \\ 0_{m,n} & & \ddots & I_n & 0_{n,n} & & & \\ -I_m & 0_{m,n} & & & & \ddots & & \vdots \\ & & & 0_{m,n} & I_m & 0_{m,n} & \ddots & \\ 0_{m,m} & & ... & & 0_{m,m} & \ddots & I_m & 0_{m,r} \\ 0_{r,m} & ... & 0_{r,n} & \widehat{\beta} & 0_{r,m} & ... & 0_{r,m} & I_r \end{bmatrix}$$

and V is a $s \times (N-s)$ matrix

$$V = \begin{bmatrix} V_{1.m,N-s} \\ V_{2.s-m,N-s} \end{bmatrix}$$

where $V_{1.m,N-s}(V_{2.s-m,N-s})$ contains the coefficients that describe the impact of the first $N-s$ rows of $\overline{X}_t$ on $c'X_t(d'X_t)$. The coefficients describing the impact of the non stationary variables on the stationary variables are zero, as are zero also the coefficients of the non-stabilizable variables on the stabilizable variables. That means the system can be divided into two subsystems, namely the stabilizable one and the non-stabilizable one. The stabilizable system comprises the first $r+m$ variables $\overline{X}_t$.

$$\overline{X}_{Rt} = \overline{A}_R\overline{X}_{Rt-1} + \overline{B}_R u_{t-1} + \overline{\mu}_{0R} + \overline{\mu}_{1R}t \ \ with \ \ \overline{A}_R = \begin{bmatrix} M_{N-s} & 0_{N-s,m} \\ V_{2.s-m,N-s} & I_m \end{bmatrix}$$

A standard optimal control problem can be solved on this stabilizable subsystem. Detectability is implicitly assured by the fact that the policymaker has already decided which variables to target (i.e., vector $c'X_t$) and has put them in the stabilizable system. The technique described in subsection 3.1 for solving the optimal control problem and deriving the coefficients ν_0 and ν_1, needed to target desired values or trends, can now be used on the stabilizable system $\overline{X}_{Rt} = \overline{A}_R\overline{X}_{Rt-1} + \overline{B}_R u_{t-1} + \overline{\mu}_{0R} + \overline{\mu}_{1R}t$

This decomposition has a very appealing feature. As emphasized above, the stabilizable state-vector $\overline{X}_{Rt}$ contains the lagged differences of all the variables, including the ones that the policymaker has chosen not to control, given the instruments to him available. This means that, although he/she cannot control the levels of $d'x_t$, he/she will still be able to control their lagged differences.

3.3 Case 3: the rank of the open-loop system cannot be augmented

Let us now consider the case in which $r_{\max} = r_{cl}$, i.e. $B \subseteq Sp(\widehat{\alpha})$. The policymaker will only be able to redefine m cointegrating vectors in a way that suits him better. To see this consider model (18) again:

$$\Delta X_t = \widehat{\alpha}\widehat{\beta}' X_{t-1} + Bu_{t-1}$$

Assuming we are using a control rule like $u_t = -FX_t$ the latter equation can be rewritten as:

$$\Delta X_t = (\widehat{\alpha}\widehat{\beta}' - BF)X_{t-1}.$$

B can be expressed as $B = a_u b_u'$, where $a_u = \widehat{\alpha}\varphi$, since $B \subseteq Sp(\widehat{\alpha})$, therefore:

$$\Delta X_t = \widehat{\alpha}(\widehat{\beta}' - \varphi b_u' F)X_{t-1} \tag{19}$$

where $\beta_c = \widehat{\beta} - F'b_u\varphi$ is the new cointegrating space. It is clearly possible to choose F so that an m-dimensional space f is included in $Sp(\beta_c)$. As Monti and Mosconi (2003) show, any F taking the general form

$$F' = (fN - \widehat{\beta}M)\overline{M}'\overline{\varphi}'\overline{b}_u'$$

where M is any $r_{cl} \times m$ full rank matrix, $\overline{M} = M(M'M)^{-1}$ ($\overline{\varphi}$ and $\overline{b}_u$ are defined in an analogous manner) and N is any $m \times m$ full rank matrix, is such that

$$\beta_c M = fN$$

$$\beta_c M_\perp = \widehat{\beta} M_\perp.$$

So, $\widehat{\beta} M$ defines the linear combinations of the original cointegration which will be modified, while $\widehat{\beta} M_\perp$ defines those that will be retained. When this control rule is applied the expression for the control instrument is

$$u_t = -\overline{b}_u \overline{\varphi}\, \overline{M}(N'f' - M'\widehat{\beta}')\widehat{X}_t \tag{20}$$

In the majority of the cases, the policy rule in equation (20) is such that u_t is non-stationary, because it will be a linear combination of $f'\widehat{X}_t$, which is stationary, but also of $M'\widehat{\beta}'\widehat{X}_t$ which is, generally, non stationary. This means that there will be no solution for a standard LQG problem with $R > 0$, only cheap control can be implemented. Nevertheless, in some very specific case it is possible to change some cointegrating relations while preserving the stationarity of the instrument. This is absolutely worth mentioning, because of its interest from the policymaking standpoint and because the two empirical exercise of section 5.1 falls in this category.

It will be possible to make m target-variables stationary with stationary instruments only if m out of the r cointegrating vectors have to relate exclusively the instruments and the target variables. For example, assume that $z_t = \begin{bmatrix} y_t & \pi_t & i_t \end{bmatrix}'$, where y_t is a measure of the output, π_t is a measure of inflation, while i_t is a short-run interest-rate and it is assumed to be the policymaker's control instrument. This model is described in detail in section 5.1, for now we will just summarize its most salient features. The estimated model is a VAR(2) cointegrated I(1) with cointegration rank 1. The cointegrating relationship has been identified as being a Fisher parity:

$$\beta_1 = \begin{bmatrix} 0 \\ -1.77 \\ 1 \end{bmatrix}$$

Let us assume that the policymaker's target is to make inflation stationary around a given mean. It's obvious that he can make inflation stationary simply by putting β_{31} to zero. Plus, he can also always add the cointegration vector $\beta_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$. The new closed-loop cointegrating matrix, generated from the control procedure is $\beta_c = \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}$. As will be shown in section 4, the new system will have a new $C = \beta_\perp(\alpha'_\perp \Gamma \beta_\perp)^{-1}\alpha'_\perp$ that has all zeros on the rows of inflation and interest rate.

4 Controlling Structural Models: The policy non-invariance problem

In this section we shall examine the consequences of implementing a new control rule in a previously estimated structural VAR. First we will analyze the new structure of the model and show how some of the features that should be policy-invariant are instead altered by policy-regime changes.

Structural VAR models have become a dominant tool for monetary policy analysis. These models are able to identify the effects of policy shocks on the economy imposing only a small number of restrictions on the system. They are also thought to be able to capture some features of the economy's functioning that are invariant to policy changes. These characteristics make them very appealing for monetary policy analysis. We will, therefore, analyze the effects of a policy change on the structural model and question whether it is really able to depict policy invariant characteristics.

Consider the following p-dimensional VAR(1), cointegrated I(1) with rank r:

$$\begin{aligned} z_t &= \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \begin{bmatrix} \Pi_{1.11} & \Pi_{1.12} \\ \Pi_{1.21} & \Pi_{1.22} \end{bmatrix} z_{t-1} + \varepsilon_t \qquad (21) \\ &= \begin{bmatrix} \Pi_{1.1} \\ \Pi_{1.2} \end{bmatrix} z_{t-1} + \varepsilon_t \end{aligned}$$

or

$$\Delta z_t = \alpha\beta' z_{t-1} + \varepsilon_t$$

where α and β are $(p \times r)$-dimensional matrices and represent, respectively, the loadings matrix and the cointegration matrix. $\mu_0 \nsubseteq Sp(\alpha)$, so it will give rise to a linear trend and $\mu_1 \subseteq Sp(\alpha)$. The subvectors x_t $(n \times 1)$ and u_t $(m \times 1)$, with $m = p - n$, respectively contain the non-policy macroeconomic variables and policy variable. $\varepsilon_t \sim i.i.d.N(0, \Omega)$. z_t. For notational simplicity we are taking into account a VAR(1) devoid of deterministic terms: remember however that every VAR(k) with deterministic terms can be expressed in the form of model (21) by using the transformation described in subsection 2.3. [7]

The strategy we use to identify the system is in line with the common-trends approach. The relationship between the reduced form residuals $\varepsilon_t \sim i.i.d.N(0, \Omega)$ and the structural form innovations $e_t = \begin{bmatrix} \varphi_t \\ v_t \end{bmatrix} \sim i.i.d.N(0, I)$ (where φ_t are permanent innovations, while ν_t are transitory ones,) is assumed to be

$$\varepsilon_t = B^{str} e_t = \begin{bmatrix} B^{str}_{1\varphi} & B^{str}_{1\nu} \\ B^{str}_{2\varphi} & B^{str}_{2\nu} \end{bmatrix} \begin{bmatrix} \varphi_t \\ \nu_t \end{bmatrix},$$

where B is a $p \times p$ non singular matrix. The VMA form of the model is:

[7] Refer to section 2.1 for a more detailed review on cointegrated VAR theory.

$$z_t = A_t + CB^{str} \sum_{i=0}^{t-1} e_{t-i} + C^*(L) B^{str} e_t.$$

where A_t is dependent on initial values and is such that $\beta' A = 0$, with $A = \lim_{t->\infty} A_t$. The CB matrix contains the long run impact coefficients for the structural innovations. The identification of the system is obtained via: orthonormality restrictions $\left(\frac{p(p+1)}{2}\right)$

$$E(e_t e_t') = B^{str-1} \Omega B^{str-1\prime} = I_p,$$

other $(p-r) \times r$ restrictions are found by assuming that only the first $s = p - r$ structural innovations, φ_t, are permanent (cumulated in X_t), while the rest, ν_t, are transitory, i.e.

$$CB^{str} U = 0_{p-r,r}.$$

Finally, the other restrictions needed can be derived from hypotheses regarding the long run effect of permanent innovations and the instantaneous impact of both transitory and permanent innovations.

The model has been estimated and identified in its closed-loop form, i.e. considering the policy instrument(s) as endogenous. In order to be able to obtain the optimal control rule we have to eliminate the old policy rule, i.e. the equation for the instrument, from the model. This operation will render the open-loop system:

$$x_t = \Pi_{11} x_{t-1} + \Pi_{12} u_{t-1} + \begin{bmatrix} B_{1\varphi}^{str} & B_{1\nu}^{str} \end{bmatrix} \begin{bmatrix} \varphi_t \\ \nu_t \end{bmatrix} \tag{22}$$

It is essential to highlight certain features of the system (22): first of all, there is no guarantee that the features of the closed-loop estimated system will continue to hold. The first issue of concern regards the roots of model (22). As highlighted before, when passing from model (21) to model (22), the system can maintain or not the same rank. If it does then the model (22) will loose as many unit roots as instruments, i.e. exactly m. In addition to this, the rest of the roots of the model also change: there is, moreover, no guarantee that all the roots of the characteristic polynomial of the VAR, $A(L) = I - \sum_{i=1}^{k} \Pi_i L^i$, will lie outside or on the unit circle. The dynamics of the system can change dramatically. Nevertheless, while the presence of unit roots in the open-loop system affects the possibility to partially or thoroughly control the system, the presence of explosive roots does not[8]. Given that our interest focuses on necessary conditions for control and the consequences of control, we will dismiss this issue and leave it for further research.

We shall now examine the consequences of implementing optimal control on a structural system. Consider again the open-loop model (22) and the optimal

[8] This result is demonstrated, in another context, in Blanchard and Kahn (1980)

control rule $u_t = -Fx_t$. After the control rule has been implemented, the closed-loop system will be:

$$\begin{bmatrix} I_n & 0 \\ F & I_m \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \begin{bmatrix} \Pi_{1.11} & \Pi_{1.12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} B_{1\varphi}^{str} & B_{1\nu}^{str} \\ 0 & B_{2\nu}^{str} \end{bmatrix} e_t.$$

The structure of model is now block recursive, because the control rule reacts instantaneously to the state variables. To return to our original structure we need to invert the matrix $\begin{bmatrix} I_n & 0 \\ F & I_m \end{bmatrix}$ and therefore[9]

$$\begin{bmatrix} x_t \\ u_t \end{bmatrix} = \Pi_1^* \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + B^{str*} e_t. \tag{23}$$

where $\Pi_1^* = \begin{bmatrix} \Pi_{1.1} \\ -F\Pi_{1.1} \end{bmatrix}$. and $B^{str*} = \begin{bmatrix} B_1^{str} & \\ -FB_{1\varphi}^{str} & B_{2\nu}^{str} \end{bmatrix}$. Model (23) is the new controlled VAR. This system will have new cointegrating and loadings matrices α^* and β^* of dimension $p \times r^*$, where r^* is the rank of the new controlled system (23). The VMA form of this new model is

$$z_t = A_t^* + C^* B^{str*} \sum_{i=0}^{t-1} e_{t-i} + C^*(L) B^{str*} e_t.$$

where $C^* = \beta_\perp^* (\alpha_\perp^{*\prime} \beta_\perp^*)^{-1} \alpha_\perp^{*\prime}$. Although, the coefficients of equations for the non-policy variables appear unchanged in the controlled system (23), the loadings and the cointegrating matrices, α^* and β^*, change. The will give rise to totally new $\alpha_\perp^*$ and $\beta_\perp^*$, and therefore to a completely different $C^* = \beta_\perp^* (\alpha_\perp^{*\prime} \beta_\perp^*)^{-1} \alpha_\perp^{*\prime}$. The empirical example reported in section 5 shows how deep and extensive these changes are.

5 Empirical Evidence

We now turn to an illustration of the above results through a couple of empirical exercises. We draw on two existing small-scale macro-models respectively for the US and the Euro area, for both of which we will derive the control rule and implement it. We will then examine the new "controlled" structural system and analyze the effects of policy changes on the structure. Both models considered fall in the category described as case 3 described in section 3.3, but the first one allows the stabilization of inflation using a stationary instrument, while in the second one it is possible to control inflation only by using a non-stationary instrument. The coefficients of these two models are reported, respectively, in Appendix B and C. The models we are using have mainly explanatory purposes and do not aspire to fully interpret the economy's functioning.

[9] Notice that $\left(\begin{bmatrix} I_n & 0 \\ F & I_m \end{bmatrix}^{-1} = \begin{bmatrix} I_n & 0 \\ -F & I_m \end{bmatrix} \right)$

There is general consensus among economists and policymakers that the central bank's main goals should be the stabilization of inflation and, possibly, the stabilization of output fluctuations. This is pursued through the use of an instrument, usually the overnight interest rate on the reserves market. As we have shown in the previous section, in many cases it is not possible to target more than one variable with one instrument. In such case, we will assume that the policymaker will choose to target inflation, which is its primary goal. If possible, we will also allow for interest rate smoothing, which is a standard feature of most optimizing monetary policy models (see, for example, Rudebusch and Svensson, 2001 or Söderström, 1999).

5.1 Model 1

The first model we refer to draws on a cointegration analysis of the model suggested by Rudebusch and Svensson (1999). It is a trivariate model comprising the following variables: the log of real GDP (y_t), inflation (π_t) measured as the annualized quarter-over-quarter difference in the GDP deflator and a 3-month money market interest rate (s_t). We use quarterly US data from 1980Q1 to 2001Q4 (for a plot, see figures 1 and 2).

Univariate analysis of the model suggests the presence of non-stationarity in both y_t and s_t. Statistical analysis for the trivariate VAR returns the following results. First of all, testing for the maximum lag, the three information criteria (AIC, BIC and HQ) agree in suggesting a maximum number of lags of 2. The trace test (Johansen,1996) for the cointegration rank indicates that the cointegration rank is 1. Tests for the deterministic components of the system suggest the presence of a constant lying out of the space of α, i.e. $\mu \nsubseteq Sp(\alpha)$, but no trend component. The estimated model is a VAR(2) cointegrated I(1) with cointegration rank r:

$$z_t = \begin{bmatrix} y_t \\ \pi_t \\ s_t \end{bmatrix} = \Pi_1 z_{t-1} + \Pi_2 z_{t-2} + \mu + \varepsilon_t \tag{24}$$

$$\Delta z_t = \alpha\beta' z_{t-1} + \Gamma_1 \Delta z_{t-1} + \mu + \varepsilon_t$$

where α and β, (3×1) matrices, are respectively the loadings matrix and the cointegrating matrix and the residuals ε_t are independent and identically distributed as a Gaussian with mean zero and variance matrix Ω, i.e. $\varepsilon_t \sim i.i.d.N(0, \Omega)$. We also define $x_t = \begin{bmatrix} y_t & \pi_t \end{bmatrix}'$, the vector containing the non-policy variables, and $u_t = s_t$, the instrument.

The single cointegrating relationship has been interpreted as a Fisher Parity on the basis of the following results. Exclusion tests performed on each of the three variables suggest that y_t can be excluded from the cointegrating relationship. When tested, this over-identifying restriction has a high significance level (see Appendix B). So, the restricted cointegrating vector is:

$$\beta = \begin{bmatrix} 0 \\ -1.7768 \\ 1 \end{bmatrix}.$$

The long-run effect of inflation on the interest rate is higher than unity, *id est* the value suggested by the Fisher relationship. This is a quite common result - see, for example, Bagliano, Golinelli, Morana (2001) - and can be explained by the fact that inflation, in the long-run, impacts on the interest rate also through an "inflation premium", that is positively related to the level of inflation. The VMA representation of this model is

$$z_t = C \sum_{i=0}^{t-1} \varepsilon_{t-i} + C^*(L)\varepsilon_t + \tau_0 + \tau_1 t$$

where the functional forms of τ_0 and τ_1 are derived in appendix A. Knowing that the relationship between reduced form and structural residuals is defined by $\varepsilon_t = B^{str} e_t$, where $e_t \sim i.i.d.N(0, I)$, one can write

$$z_t = CB^{str} \sum_{i=0}^{t-1} e_{t-i} + C^*(L)B^{str}B^{str-1}\varepsilon_t + \tau_0 + \tau_1 t$$

The SVAR analysis returns the following results. B^{str} is a 3×3 symmetric matrix, so $\frac{3\times(3+1)}{2}$ restrictions are needed to identify the matrix: 3 of them are derived from the orthonormality of the structural residuals. Since the model is I(1) with rank 1, it will have 2 common trends, the condition

$$\alpha'_{\perp} B^{str} U = 0_{2,1},$$

that allows to identify the two permanent innovations $\varphi_t = \begin{bmatrix} \varphi_{1t} & \varphi_{2t} \end{bmatrix}'$ from the transitory one ν_t, will convey other 2 restrictions. One more restriction is needed to obtain identification. If we assume that φ_{1t} is the real permanent innovation, while φ_{2t} is the nominal permanent innovation, the most natural and most commonly used restriction is to assume that the nominal permanent innovation φ_{2t} will not have a permanent effect on the real variables, i.e. on y_t. This will return a long-run impact matrix CB^{str} of the following form:

$$CB^{str} = \begin{bmatrix} [CB^{str}]_{11} & 0 & 0 \\ [CB^{str}]_{21} & [CB^{str}]_{22} & 0 \\ [CB^{str}]_{31} & [CB^{str}]_{32} & 0 \end{bmatrix}.$$

The system we have estimated includes the estimates of the "historical" policy rule. We now want to define a new policy rule, obtained with optimal control techniques. We will follow the procedure described in subsection 2.3 to rewrite model (24) as a first order homogenous linear system, first eliminating the deterministic components then defining the companion form system (6×1)

$$\widetilde{Z_t} = \widetilde{\Pi}_1 \widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t \;\; or \;\; \Delta\widetilde{Z_t} = \widetilde{\alpha}\widetilde{\beta}' \widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t$$

where

$$\widetilde{Z_t} = \begin{bmatrix} \overline{z}_t \\ \overline{z}_{t-1} \end{bmatrix} \ , \ \widetilde{\Pi}_1 = \begin{bmatrix} \Pi_1 & \Pi_2 \\ I_p & 0 \end{bmatrix} \ , \ \widetilde{\varepsilon}_t = \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix}.$$

and $\overline{z}_t = z_t - \tau_0 - \tau_1 t$. This model retains the same number of unit roots, i.e. 2, of the system it derives from: therefore, it has rank 4. Removing the equation for the control instrument at time t u_t, the system becomes

$$X_t = AX_{t-1} + Bu_{t-1} + \widehat{B}^{str}\hat{e}_t \ \ or \ \ \Delta X_t = \widehat{\alpha}\widehat{\beta}' X_{t-1} + Bu_{t-1} + \widehat{B}^{str}\hat{e}_t \quad (25)$$

where $X_t = \begin{bmatrix} x_t & x_{t-1} & u_{t-1} \end{bmatrix}'$ is (5×1). Model (25) has cointegration rank 4 and only one unit root.

The first step of our analysis is to understand to which extent the system can be controlled, following the "roadmap" defined in section 3. Since B lies in the space of $\widehat{\alpha}$, i.e. $B \subseteq Sp(\widehat{\alpha})$, the rank of the open-loop system cannot be augmented (this system falls in the third of the three cases described in section 3). Therefore, the only possibility the policymaker has is to redefine the cointegrating vector in a manner that suits him better. In this case, generally, he/she will only be able to implement cheap control, i.e. a type of linear-quadratic control where no cost is assigned to the instrument. Nonetheless, as we have mentioned before, this case is special, because the form of the cointegrating vector allows for a stationary control rule to be implemented and, therefore a standard optimal control problem can be solved.

We are therefore assuming that the policymaker is following a strict inflation targeting strategy with interest rate smoothing and that his target for the annualized quarterly inflation rate is 2%. As highlighted before, the solution to this problem is a policy rule of the form $u_t = -FX_t + \nu_0$, where F and ν_0 can be found separately, as already emphasized in section 3. We will first recover F that renders the inflation rate stationary around its mean and later we will define ν_0, so as to make it stationary around a desired value, different from its mean.

The policymaker's loss function can be expressed as:

$$E_t \left(\sum_{i=0}^{\infty} L_{t+i} \right)$$

where $L_{t+i} = Y'_{t+i} K Y_{t+i}$ and $Y_t = HX_t + Ju_t$, with

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \ , \ J = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \ K = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.2 \end{bmatrix}$$

and $Q = H'KH, W = H'KJ, R = J'KJ$. The control problem can be solved by solving iteratively the Differential Riccati Equation to convergence. Once the $\widehat{P}$ Riccati matrix is found, the control rule, expressed in terms of the open-loop system can be recovered:

$$u_t = -FX_t = -\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} y_t & \pi_t & y_{t-1} & \pi_{t-1} & s_{t-1} \end{bmatrix}'$$

where

$$F = (R + B'\widehat{P}B)^{-1}(B'\widehat{P}A + W').$$

and F_1 is a (2×1) matrix containing the coefficients that describe the instantaneous relation between the instrument and the non-policy variables, while F_2 is a (3×1) matrix describing the reaction of the policy instrument to the lags of all variables. All the parameters of the system, the control rule and the parameters of the new system obtained with this control rule are reported in Appendix B. The companion form of the controlled system is:

$$X_t = (A - BF)X_{t-1}$$

So, the controlled system in its VAR form is:

$$\begin{aligned} \begin{bmatrix} I_n & 0 \\ F_1 & I_m \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} &= \begin{bmatrix} \Pi_{1.11} \quad \Pi_{1.12} \\ -F_2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \\ &+ \begin{bmatrix} \Pi_{2.11} \quad \Pi_{2.12} \\ 0_{1,3} \end{bmatrix} \begin{bmatrix} x_{t-2} \\ u_{t-2} \end{bmatrix} + \begin{bmatrix} B_{1\varphi} & B_{1\nu} \\ 0 & B_{2\nu} \end{bmatrix} e_t. \end{aligned} \tag{26}$$

The structuralisation of this model is clearly different from the one previously estimated. To return to the latter we will premultiply model (26) by the inverse of $\begin{bmatrix} I_n & 0 \\ F_1 & I_m \end{bmatrix}$. Defining, $\Pi_{1.1} = \begin{bmatrix} \Pi_{1.11} & \Pi_{1.12} \end{bmatrix}$ and $\Pi_{2.1} = \begin{bmatrix} \Pi_{2.11} & \Pi_{2.12} \end{bmatrix}$ one can write

$$\begin{aligned} z_t &= \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \Pi_1^* z_{t-1} + \Pi_2^* z_{t-2} + \mu^{new} + B^{str*} e_t = \\ &= \begin{bmatrix} \Pi_{1.1} \\ -F_1\Pi_{1.1} - F_2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} \Pi_{2.1} \\ -F_1\Pi_{2.1} \end{bmatrix} \begin{bmatrix} x_{t-2} \\ u_{t-2} \end{bmatrix} + \\ &+ \begin{bmatrix} \mu_x \\ -F_1\mu_x \end{bmatrix} + B^{str*} e_t. \end{aligned} \tag{27}$$

It is straightforward to show that the new $\Pi^* = \alpha^* \beta^{*\prime} = -I + \Pi_1^* + \Pi_2^*$ has rank 2. The new cointegrating vector is:

$$\beta^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the new loadings matrix can be easily found as $\alpha^* = \Pi^* \overline{\beta^*}$, where is $\overline{\beta}^*$ defined as $\overline{\beta}^* = \beta^*(\beta^{*\prime}\beta^*)^{-1}$. The VMA form of this model is:

$$z_t = C^* \sum_{i=0}^{t-1} \varepsilon_{t-i} + C^*(L)\varepsilon_t + \tau_0^{new}$$

where $C^* = \begin{bmatrix} C_{11}^* & C_{21}^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It is easy to notice, given the form of C^*, that the inflation and the interest rate will both be stationary. As proved in Appendix A, when $\mu_0 \nsubseteq Sp(\alpha)$ and $\mu_1 = 0$, $\tau_0^{new} = C^* X_0 - \alpha^*(\beta^{*\prime}\alpha^*)^{-1}(\beta^{*\prime}\alpha^*)^{-1}\beta^{*\prime}\mu^{new}$.

The next step of the control procedure involves the recovery of ν_0. Equation (27) displays the model in which the control rule $u_t = -FX_t$. If we were to implement the rule $u_t = -FX_t + \nu_0$, the model would be:

$$\begin{aligned} z_t^{ctr} &= \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \Pi_1^* z_{t-1} + \Pi_2^* z_{t-2} + \mu^{ctr} + B^{str*} e_t = \\ &= \begin{bmatrix} \Pi_{1.1} \\ -F_1 \Pi_{1.1} - F_2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} \Pi_{2.1} \\ -F_1 \Pi_{2.1} \end{bmatrix} \begin{bmatrix} x_{t-2} \\ u_{t-2} \end{bmatrix} + \\ &\quad + \begin{bmatrix} \mu_x \\ -F_1 \mu_x + \nu_0 \end{bmatrix} + B^{str*} e_t. \end{aligned}$$

B^{str*} is obtained as:

$$B^{str*} = \begin{bmatrix} I & 0 \\ -F_1 & I \end{bmatrix}^{-1} \begin{bmatrix} \widehat{B}^{str} \\ 0 \quad B_{33}^{str} \end{bmatrix}$$

Now, $\lim_{t\to\infty} E(z_t^{ctr}|z_0) = \tau_0^{ctr} = C^* X_0 - \alpha^* (\beta^{*\prime} \alpha^*)^{-1} (\beta^{*\prime} \alpha^*)^{-1} \beta^{*\prime} \mu^{ctr}$. Assume that the policymaker's goal is to drive the inflation rate to a 2% mean, i.e. to impose $\tau_{0.2}^{ctr} = 0.02$. With some algebra it is straightforward to prove that, in order to obtain $\tau_{0.2}^{ctr} = 0.02$, one has to impose $\nu_0 = 0.038811$.

Figures 3 displays the simulation of the controlled system for the deterministic case. The main purpose of the empirical exercise was to verify the effectiveness of the control rule: nonetheless, always bearing in mind that this is a very small and stylized model, some economic intuitions can be recovered. First of all, Figure 3 shows that inflation can be made stationary and driven to its target in 15 to 20 quarters. This figure also show how the Fisher parity holds in the long-run, allowing ex post for a positive and constant real interest rate $\rho_t = \pi_t - s_t$. Figures 4, 5 and 6 present the three variables of the simulated system separately. Comparing these figures, it is straightforward to see that inflation and the interest rate are stationary, while y_t maintains the unit root and allows for stochastic trends in addition to its deterministic trend. Looking at Figures 5 and 6 it is easy to notice that the interest rate is definitely smoother than the inflation rate, due to the presence of an interest rate smoothing term in the loss function.

We have shown that the control rule we have implemented succeeds in making inflation stationary around the desired mean. But what is the impact of the control rule on the structural system? Appendix B reports the coefficients of the estimated long-run impact matrix CB^{str} and the coefficients of the long-run impact matrix $C^* B^{str*}$ obtained after the new policy rule is implemented. The two matrices have very different features: in particular, in the $C^* B^{str*}$, both inflation and interest rate are stationary and, hence, do not have any entry in the long-run impact matrix of the controlled system. The structural features of the system, i.e. the ones that should be invariant to changes in the policy regime, are the following:

- The number and designation of the permanent and transitory structural innovations

- The long-run neutrality of money, modeled by imposing that only the real permanent structural innovation has an impact on the real variable (GDP)
- The steady state of the real variables.

Only the first feature is maintained in the controlled system, while the others are lost. This obviously undermines the model, since the long-run neutrality of money hypothesis is entrenched wisdom in economic theory: a valid model should maintain this feature.

5.2 Model 2

The second model we shall consider is based on Brand and Cassola (2002). In this study we use quarterly data on broad money (M3), GDP, the GDP deflator and interest rates from 1980Q1 through 2001Q4. In particular the following five variables will be used to represent the economy: logs of real GDP (y_t), inflation (π_t) measured as the quarter over-quarter change in the logs of the GDP deflator. Short-term rates (s_t) are 3-month money market interest rates and long-term interest rates (l_t) are 10 year government bond yields or close substitutes.

Univariate analysis of these six variables suggests that $(m-p)_t$ and y_t can be characterized as I(1) processes with positive drift and s_t, π_t and l_t as I(1) processes without drift. These variables seem to be well represented by a cointegrated I(1) VAR(2) with rank 3, constant component $\mu_0 \nsubseteq Sp(\alpha)$ and a trend component $\mu_1 \subseteq Sp(\alpha)$ (so that it will not give rise to a quadratic trend):

$$z_t = \begin{bmatrix} (m-p)_t \\ \pi_t \\ l_t \\ y_t \\ s_t \end{bmatrix} = \Pi_1 z_{t-1} + \Pi_2 z_{t-2} + \mu_0 + \alpha b_1' t + \varepsilon_t \tag{28}$$

$$\Delta z_t = \alpha\beta' z_{t-1} + \Gamma_1 \Delta z_{t-1} + \mu_0 + \alpha b_1' t + \varepsilon_t$$

where α and β, (5×3) matrices, are respectively the loadings matrix and the cointegrating matrix and the residuals ε_t are independent and identically distributed as a Gaussian with mean zero and variance matrix Ω, i.e. $\varepsilon_t \sim i.i.d.N(0, \Omega)$. We also define $x_t = \begin{bmatrix} (m-p)_t & \pi_t & l_t & y_t \end{bmatrix}'$, the vector containing the non-policy variables, and $u_t = s_t$, the instrument. The three cointegrating relations have been identified in the following way:

$$\beta = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -0.5739 & 0 \\ -1 & 0 & -0.1412 \\ 0 & 0 & 1 \end{bmatrix}$$

The first relationship β_1 can be interpreted as a long-run money demand function, β_2 is a Fisher Parity relationship, while the third cointegrating vector can be viewed as a long-run policy function. A more detailed description of the model is presented in Appendix C.

The following step of our analysis is to verify to what degree this system can be controlled. To do so, we first have to define the system in the open-loop form, following the procedure described in section 2.3 and obtain

$$X_t = AX_{t-1} + Bu_{t-1} + \widehat{B}^{str}\hat{e}_t \ \ or \ \ \Delta X_t = \widehat{\alpha}\widehat{\beta}' X_{t-1} + Bu_{t-1} + \widehat{B}^{str}\hat{e}_t$$

where $X_t = \begin{bmatrix} x_t & x_{t-1} & u_{t-1} \end{bmatrix}'$ is (9×1). This model has cointegration rank 8 and only one unit root. Also in this second example B lies in the space of $\widehat{\alpha}$, i.e. $B \subseteq Sp(\widehat{\alpha})$, so the rank of the open-loop system cannot be augmented. In addition to this, there is no cointegrating relationship involving exclusively the instrument and and the target variable, which is assumed to be inflation: in line with the findings of section 3, it will be possible to make inflation stationary only rendering the instrument non-stationary. This implies that only cheap control (i.e., $R = 0$) can be implemented on this model. Let us assume our goal is to make inflation stationary around the value 2%: it has been proven in section 3 that it is possible to find first the matrix F, such that $u_t = -FX_t$ make inflation stationary around its own trend, and then define ν_0 and ν_1 that will drive it to be stationary around the desired value 2%. We shall now find F. So, the policymaker's loss function can be expressed as:

$$E_t\left(\sum_{i=0}^{\infty} L_{t+i}\right)$$

where $L_{t+i} = Y_{t+i}'KY_{t+i}$ and $Y_t = HX_t$, with

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \ , \ J = [0] \ , \ K = [1]$$

and $Q = H'KH, W = H'KJ = 0$ and $R = J'KJ = 0$. The control problem can be solved by solving iteratively the Differential Riccati Equation to convergence. Once the $\widehat{P}$ Riccati matrix is found, the control rule, expressed in terms of the open-loop system can be recovered:

$$u_t = -FX_t = -\begin{bmatrix} F_1 & F_2 \end{bmatrix}\begin{bmatrix} x_t & x_{t-1} & u_{t-1} \end{bmatrix}'$$

where

$$F = (B'\widehat{P}B)^{-1}(B'\widehat{P}A + W').$$

and F_1 is a (4×1) matrix containing the coefficients that describe the instantaneous relation between the instrument and the non-policy variables, while F_2 is a (5×1) matrix describing the reaction of the policy instrument to the lags of all variables. All the parameters of the system, the control rule and the parameters of the new system obtained with this control rule are reported in Appendix C. The companion form of the controlled system is:

$$X_t = (A - BF)X_{t-1}$$

The new controlled VAR(2) can be obtained exactly as in the previous example (see equations (26) and (27)) and it will be

$$
\begin{aligned}
z_t &= \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \Pi_1^* z_{t-1} + \Pi_2^* z_{t-2} + \mu_0^{new} + \mu_1^{new} t + B^{str*} e_t = \\
&= \begin{bmatrix} \Pi_{1.1} \\ -F_1 \Pi_{1.1} - F_2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} \Pi_{2.1} \\ -F_1 \Pi_{2.1} \end{bmatrix} \begin{bmatrix} x_{t-2} \\ u_{t-2} \end{bmatrix} + \\
&\quad + \begin{bmatrix} \mu_{0x} \\ -F_1 \mu_{0x} \end{bmatrix} + \begin{bmatrix} \mu_{1x} \\ -F_1 \mu_{1x} \end{bmatrix} t + B^{str*} e_t.
\end{aligned}
$$

It is straightforward to show that the new $\Pi^* = \alpha^* \beta^{*\prime} = -I + \Pi_1^* + \Pi_2^*$ has rank 2 and that μ_1^{new} still lies in the space of $\widehat{\alpha}$ (no quadratic trend). The new cointegrating vector is:

$$
\beta^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -0.1412 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

and the new loadings matrix can be easily found as $\alpha^* = \Pi^* \overline{\beta^*}$, where is $\overline{\beta}^*$ defined as $\overline{\beta}^* = \beta^* (\beta^{*\prime} \beta^*)^{-1}$. The VMA form of this model is:

$$
z_t = C^* \sum_{i=0}^{t-1} \varepsilon_{t-i} + C_1^*(L) \varepsilon_t + \tau_0^{new} + \tau_1^{new} t \tag{29}
$$

where $C^* = \begin{bmatrix} C_{11}^* & C_{12}^* & C_{13}^* & C_{14}^* & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_{41}^* & C_{42}^* & C_{43}^* & C_{44}^* & 0 \\ C_{51}^* & C_{52}^* & C_{53}^* & C_{54}^* & 0 \end{bmatrix}$. It is easy to notice, given the form of C^*, that the inflation and the long-term interest rate will both be stationary.

Now that we have obtained model (29), we can derive ν_0 and ν_1. We know, from Appendix A, that $\tau_0^{new} = C^* X_0 - \alpha^* (\beta^{*\prime} \alpha^*)^{-1} (\beta^{*\prime} \alpha^*)^{-1} \left(\beta^{*\prime} \mu^{new} + Ktr \right)$ and that $\tau_1^{new} = C \mu_0^{new} - \alpha (\beta' \alpha)^{-1} (\beta' \alpha)^{-1} \beta' \mu_1^{new}$, but if we were to implement the rule $u_t = -F X_t + \nu_0 + \nu_1 t$, the model would be:

$$
\begin{aligned}
z_t^{ctr} &= \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \Pi_1^* z_{t-1} + \Pi_2^* z_{t-2} + \mu^{ctr} + B^{str*} e_t = \\
&= \begin{bmatrix} \Pi_{1.1} \\ -F_1 \Pi_{1.1} - F_2 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} \Pi_{2.1} \\ -F_1 \Pi_{2.1} \end{bmatrix} \begin{bmatrix} x_{t-2} \\ u_{t-2} \end{bmatrix} + \\
&\quad + \begin{bmatrix} \mu_{0x} \\ -F_1 \mu_{0x} + \nu_0 \end{bmatrix} + \begin{bmatrix} \mu_{1x} \\ -F_1 \mu_{1x} + \nu_1 \end{bmatrix} t + B^{str*} e_t.
\end{aligned}
$$

B^{str*} is build as in the previous example. Now, $\lim_{t \to \infty} E(z_t^{ctr} | z_0) = \tau_0^{ctr} + \lim_{t \to \infty} \tau_1^{ctr} t$, where τ_0^{ctr} and τ_1^{ctr} are as:

$$
\tau_1^{ctr} = C^* \mu_0^{ctr} - \alpha (\beta' \alpha)^{-1} (\beta' \alpha)^{-1} \beta' \mu_1^{ctr}
$$

$$
\tau_0^{ctr} = C^* X_0 - \alpha^* (\beta^{*\prime} \alpha^*)^{-1} (\beta^{*\prime} \alpha^*)^{-1} \left(\beta^{*\prime} \mu^{ctr} + Ktr \right)
$$

The policymaker's goal is to drive the inflation rate to a 2% mean, i.e. he wants to impose $\tau^{ctr}_{0.2} = 0.02$ and $\tau^{ctr}_{1.2} = 0$. Remembering that the second row of C^* is all filled with zeros, with some algebra it is quite simple to define ν_0 and ν_1 that satisfy the policymaker's needs (the values are reported in Appendix C).

Figures 7-9 show the simulated controlled VAR. As expected, it is possible to drive inflation to be stationary around 2%, but the instrument will necessarily be non-stationary (Figures 7 and 8). Figure 9 shows the pattern of real GDP and real money balances in the controlled system.

What can be said regarding the new structure of the model is analogous to what was said in the previous section regarding model 1. Appendix C reports the coefficients of the estimated long-run impact matrix CB^{str} and the coefficients of the long-run impact matrix C^*B^{str*} obtained after the new policy rule is implemented. The two matrices have very different features: in particular, in the C^*B^{str*}, both inflation and the long-run interest rate are stationary and, hence, do not have any entry in the long-run impact matrix of the controlled system. The issues regarding the structure of the model, presented in Section 4, are evident: the property of long-run neutrality of money is lost and the steady state of the real variables is affected by policy.

6 Conclusions

The scope of this paper is very broad: its main accomplishment is to unveil the most critical issues that arise when implementing optimal control techniques in Structural Cointegrated VAR models and, in particular, in Stochastic Common Trends Models. The former techniques and the latter models are extensively exploited in the economic literature regarding monetary policy analysis, but these issues have been apparently neglected.

Our approach to such a broad matter is the following. We first analyze the implications and the effects of the presence of unit roots in the model on the policymaker's ability to control the system, partially or completely. Based on the results obtained in Monti and Mosconi (2003), we report the regularity conditions that the system has to meet for the policymaker to be able to stabilize it, partially or thoroughly. For the different cases we then derive the optimizing control technique to be adopted. In section 5 we present two empirical examples. The first one is a small scale macro-model, comprising inflation rate, the GDP and the interest rate, while the second one is a five-variable model including real money balances, inflation long-run and short run interest rate and real output.

The second part of our analysis regards the impact on the structural form of changes in the control rule. The structural form of the model is, by definition, built to capture features of the model that are supposed to be invariant to changes in the policy regime. Our findings contradict this assumption. In order to control the system it is obvious that the control rule has to influence the dynamics of the non-policy variables: what surprises is to see that all coefficients of the system can be affected by the policy changes. This means that features of the system that are assumed to be invariant to policy are instead affected by policy changes. Both models in section 5 have been identified also by the long-run neutrality on money on the real variables. This latter assumption is entrenched wisdom in recent macroeconomic theory and is generally built in as a feature in most macroeconomic models. In both models in section 5, nonetheless, the feature of long-run neutrality of money disappears. This paper acknowledges this shortcoming. Nevertheless, since the policy non-invariance property of certain features is essential for the soundness of the SVAR approach, the following step in this research is, if possible, the definition of conditions on the VAR coefficients that allows policy non-invariance of some of the VMA representation coefficients.

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A Derivation of the Functional Form of the Deterministic Components in the VMA Representation of a VAR

Consider the following p-dimensional VAR(1), cointegrated I(1) with cointegration rank r:

$$X_t = \Pi_1 X_{t-1} + \mu_0 + \mu_1 t + \varepsilon_t \tag{A.1}$$

where $\varepsilon_t \sim N(0, \Omega)$. There is no loss in generality in considering a VAR with a single lag, since every VAR(k), where k is the maximum number of lags, can be rewritten as a VAR(1) using the so-called Companion Form. The VECM form of (A.1) is

$$\Delta X_t = \alpha\beta' X_{t-1} + \mu_0 + \mu_1 t + \varepsilon_t$$

where α and β, (pxr) matrices, are respectively the loadings matrix and the cointegration matrix. Following Johansen(1996) it is possible to write model (A1) in the Moving Average Representation (MAR) form, i.e. only in terms the shocks and the deterministic components:

$$X_t = C\sum_{i=0}^{t-1} \varepsilon_{t-i} + C^*(L)\varepsilon_t + \tau_0 + \tau_1 t + \tau_2 t^2$$

Our goal is to derive the functional form of τ_0,τ_1 and τ_2 .

The solution of this model with initial value X_0 can be written as

$$X_t = (I + \alpha\beta')^t X_0 + \sum_{i=0}^{t-1} (I + \alpha\beta')^i [\mu_0 + \mu_1(t-i) + \varepsilon_{t-i}] \tag{A.2}$$

We are interested in defining the long-run coefficients for the deterministic part in the most general way, i.e. for μ_0 and μ_1 both possibly $\nsubseteq Sp(\alpha)$.To do so, we shall use the following relations

$$\alpha(\beta'\alpha)^{-1}\beta' + \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp = I \tag{A.3}$$

$$\alpha'_\perp(I + \alpha\beta')^t = \alpha'_\perp$$

$$\beta'(I + \alpha\beta')^t = (I_r + \beta'\alpha)^t\beta'$$

Using the first relation, the expected value of X_t given the initial vales X_0, i.e. $E_t(X_t|X_0)$, can be rewritten as:

$$E_t(X_t|X_0) = \alpha(\beta'\alpha)^{-1}E_t(\beta' X_t|X_0) + \beta_\perp(\alpha'_\perp\beta_\perp)_{-1}E_t(\alpha'_\perp X_t|X_0). \tag{A.4}$$

Now using equation (A.2) to define X_t and using the relations in (A.3), we can write the two components $E_t(\alpha'_\perp X_t|X_0)$ and $E_t(\beta' X_t|X_0)$ respectively as

$$
\begin{aligned}
E_t(\alpha'_\perp X_t|X_0) &= \alpha'_\perp X_0 + \sum_{i=0}^{t-1} \alpha'_\perp[\mu_0 + \mu_1(t-i)] = \qquad (A.5)\\
&= \alpha'_\perp X_0 + \alpha'_\perp \mu_0 t + \alpha'_\perp \mu_1 \frac{t(t+1)}{2} = \\
&= \alpha'_\perp X_0 + (\alpha'_\perp \mu_0 + \alpha'_\perp \mu_1/2)t + \alpha'_\perp \mu_1 \frac{t^2}{2}
\end{aligned}
$$

and

$$
E_t(\beta' X_t|X_0) = (I_r + \beta'\alpha)^t \beta' X_0 + \sum_{i=0}^{t-1} (I_r + \beta'\alpha)^i \beta'[\mu_0 + \mu_1(t-i)]
$$

The model is defined as being I(1), therefore

$$
\left| eig(I_r + \beta'\alpha) \right| < 1
$$

. This means that $\lim_{t\to\infty}(I_r + \beta'\alpha)^t = 0$ and that

$$
\sum_{i=0}^{\infty} (I_r + \beta'\alpha)^i = (I_r - (I_r + \beta'\alpha))^{-1} = (-\beta'\alpha)^{-1}
$$

. The limit of $E_t(\beta' X_t|X_0)$ for $t \to \infty$ can be expressed as

$$
\begin{aligned}
\lim_{t\to\infty} E_t(\beta' X_t|X_0) &= (-\beta'\alpha)^{-1}\beta'\mu_0 + \lim_{t\to\infty} \sum_{i=0}^{t-1} (I_r + \beta'\alpha)^i \beta' \mu_1 (t-i) \qquad (A.6)\\
&= (-\beta'\alpha)^{-1}\beta'\mu_0 + \lim_{t\to\infty} (-\beta'\alpha)^{-1}\beta'\mu_1 t + Ktr
\end{aligned}
$$

where $Ktr = \sum_{i=0}^{\infty} (I_r + \beta'\alpha)^i \beta' \mu_1(-i)$ is a constant value. It is easy to see it will converge, considering that $(I_r + \beta'\alpha)^t$ has a higher "speed of convergence" compared to the speed at which t diverges. Now, using equation (A.4), it is possible to define $\lim_{t\to\infty} E_t(X_t|X_0)$ combining equations (A.5) and (A.6), i.e.

$$
\begin{aligned}
\lim_{t\to\infty} E_t(X_t|X_0) &= \beta_\perp(\alpha'_\perp\beta_\perp)_{-1} \lim_{t\to\infty} E_t(\alpha'_\perp X_t|X_0) + \alpha(\beta'\alpha)^{-1} \lim_{t\to\infty} E_t(\beta' X_t|X_0) = \\
& CX_0 + \lim_{t\to\infty}(C\mu_0 + C\mu_1/2)t + \lim_{t\to\infty} C\mu_1 \frac{t^2}{2} + \qquad (A.7)\\
& + \alpha(\beta'\alpha)^{-1}[(-\beta'\alpha)^{-1}\beta'\mu_0 + Ktr + \lim_{t\to\infty}(-\beta'\alpha)^{-1}\beta'\mu_1 t]
\end{aligned}
$$

where $C = \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp$.

We can now derive the form of τ_0,τ_1 and τ_2 in the MAR .

- $\tau_0 = CX_0 - \alpha(\beta'\alpha)^{-1}(\beta'\alpha)^{-1}[\beta'\mu_0 + Ktr]$

- $\tau_1 = (C\mu_0 + C\mu_1/2) - \alpha(\beta'\alpha)^{-1}(\beta'\alpha)^{-1}\beta'\mu_1$
- $\tau_2 = C\mu_1/2$

The above definitions are meant to be as general as possible, comprising the possibility that both μ_0 and $\mu_1 \not\subseteq Sp(\alpha)$

B Model 1

The dataset we refer to is the FRED database, available on the website of the Federal Reserve Bank of St. Louis. Refer to section 5.1 for the detail description of the model and the procedures followed. The estimated coefficients for the trivariate VAR(2) on the process

$$z_t = \begin{bmatrix} y_t \\ \pi_t \\ s_t \end{bmatrix} = \Pi_1 z_{t-1} + \Pi_2 z_{t-2} + \mu + B^{str} e_t$$

$$\Delta z_t = \alpha\beta' z_{t-1} + \Gamma_1 \Delta z_{t-1} + \mu + B^{str} e_t,$$

where $e_t \sim N(0, I)$, are:

$$\Gamma_1 = \begin{bmatrix} 0.2902 & -0.0391 & -0.1154 \\ 0.0900 & -0.1757 & 0.0168 \\ 0.3942 & -0.3151 & 0.0855 \end{bmatrix}$$

and

$$\mu = \begin{bmatrix} 0.0052 \\ -0.0023 \\ -0.0009 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} 0.0101 \\ 0.0895 \\ -0.2539 \end{bmatrix} \quad and \;\; \beta = \begin{bmatrix} 0 \\ -1.7768 \\ 1 \end{bmatrix}.$$

A LR is used to test the over-identifying restriction on the cointegrating vector. The statistics is distributed as a $\chi^2(1)$, i.e. with one degree of freedom, and returns a value of 1.92620, corresponding to a significance level of 0.16518: therefore, the restriction cannot be rejected.

B^{str}, i.e. the matrix that describes the relations existing between the reduced form residuals and the structural innovations, is identified by assuming that $CB^{str}U = 0_{p,r}$ and that the nominal structural innovation will not have a permanent impact on real variables, i.e. $[CB]_{12} = 0$. We obtain

$$B^{str} = \begin{bmatrix} 0.00618 & -0.00052 & -0.00023 \\ 0.00191 & 0.00816 & -0.00201 \\ 0.0036 & 0.00221 & 0.0057 \end{bmatrix}.$$

The long-run impact matrix is:

$$CB^{str} = \begin{bmatrix} 0.0095 & 0 & 0 \\ 0.0029 & 0.0049 & 0 \\ 0.0052 & 0.0087 & 0 \end{bmatrix}$$

The solution of the control problem, derived in section 5.1 returns the following law for the instrument:

$$u_t = -FX_t + \nu$$

where

$$F = \begin{bmatrix} 0.11777 & 0.80174 & -0.11777 & 0.17561 & -0.65732 \end{bmatrix}$$

$$\nu = 0.038811.$$

The new VECM system, obtained by implementing the control rule above, has the following characteristics:

$$\alpha^* = \begin{bmatrix} -0.0179 & 0.0101 \\ -0.1590 & 0.0895 \\ -0.8477 & -0.4156 \end{bmatrix} \textit{ and } \beta^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C^* = \begin{bmatrix} 1.3890 & -0.1567 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The coefficients of the VAR are:

$$\Pi_1^* = \begin{bmatrix} 1.2902 & -0.0570 & 0.1255 \\ 0.0900 & 0.6653 & 0.1063 \\ -0.1063 & -0.7023 & 0.5573 \end{bmatrix}, \ \Pi_2^* = \begin{bmatrix} -0.2902 & 0.0391 & -0.1154 \\ -0.0900 & 0.1757 & -0.0168 \\ 0.1063 & -0.1455 & 0.0271 \end{bmatrix}$$

and

$$B^{str*} = \begin{bmatrix} 0.0062 & -0.0005 & -0.0002 \\ 0.0019 & 0.0082 & -0.0020 \\ -0.0023 & -0.0065 & 0.0073 \end{bmatrix}$$

So, the long-run impact matrix is

$$C^* B^{str*} = \begin{bmatrix} 0.0083 & -0.0020 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

C Model 2

The estimated α and β for the five-variable VAR(2) on the process treated in section 5.2

$$\Delta z_t = \alpha\beta' z_{t-1} + \Gamma_1 \Delta z_{t-1} + \mu_0 + \mu_1 t + B^{str} e_t,$$

where $e_t \sim N(0, I)$, are:

$$\alpha = \begin{bmatrix} -0.1298\ (0.031) & 0\ (0) & 0\ (0) \\ 0\ (0) & -0.4051\ (0.11) & 0\ (0) \\ 0.0283\ (0.0106) & 0\ (0) & -0.1939\ (0.048) \\ 0.1228\ (0.056) & -1.058\ (0.347) & -0.986\ (0.335) \\ 0.0281\ (0.012) & -0.3456\ (0.064) & -0.5308\ (0.069) \end{bmatrix}$$

$$\beta = \begin{bmatrix} 1\ (0) & 0\ (0) & 0\ (0) \\ 0\ (0) & 1\ (0) & -1\ (0) \\ 0\ (0) & -0.574\ (0.0554) & 0\ (0) \\ -1\ (0) & 0\ (0) & -0.141\ (0.0119) \\ 0\ (0) & 0\ (0) & 1\ (0) \\ -0.00239\ (8.48 \times 10^{-5}) & 0\ (0) & 0.00095\ (7.44 \times 10^{-5}) \end{bmatrix}$$

The standard errors on the α and β coefficients reported in the parenthesis. The other estimated coefficients are

$$\Gamma_1 = \begin{bmatrix} 0.5535 & -0.0169 & -0.6063 & -0.0598 & -0.4166 \\ 0.0023 & -0.3579 & 0.0275 & -0.1004 & 0.5989 \\ 0.0244 & -0.0644 & 0.4154 & 0.0002 & 0.1410 \\ 0.0846 & 0.3749 & 0.0365 & 0.0551 & 0.5477 \\ 0.0469 & -0.0655 & -0.0120 & -0.0687 & 0.5536 \end{bmatrix}$$

and

$$\mu_0 = [-0.058\ \ -0.0011\ \ -0.3976\ \ -2.0304\ \ -1.113]'.$$

Brand and Cassola (2002) draw on their earlier model, described in Brand and Cassola (2000), to develop this new model. The data used is the same, while the sample period is now 1980Q1-2001Q3. The over-identifying restrictions are tested within PCGive 10.3, copyright by J. A. Doornik, using LR test, whose statistics is distributed as a chi-square: the value of the statistics is 9.7374. The software identifies 10 over-identifying restrictions, so the distribution is a chi-square with 10 degrees of freedom, $\chi^2(10)$. Therefore these restrictions have a significance level of 0.4638 and they cannot be rejected.

The first cointegration relationship can be interpreted as a long run money demand function with a "velocity specification". As a time trend could not be excluded from the cointegration space, the historical decline in M3 income velocity is captured by the inclusion of a trend in the long-run money demand relation, while imposing an income elasticity of one on money demand. The second cointegrating relationship is a Fisher relationship and remains unchanged from Brand and Cassola (2000) to Brand and Cassola (2002). The authors find a different specification for the third cointegrating vector, which links (ex-post) real short term rates with deviations of output from a linear time trend. This yields a long-run monetary policy reaction function which specifies that real-short term rates are set in response to deviation of output from a linear trend, which has some similarities with a Taylor rule.

The matrix B^{str}, i.e. the matrix that describes the relations existing between the reduced form residuals and the structural innovations, is identified by assuming that $CB^{str}U = 0_{5,3}$ and that the nominal structural innovation will not have a permanent impact on real variables. We obtain

$$B^{str} = \begin{bmatrix} -0.00218 & -0.00060 & 0.00091 & 0.00248 & 0 \\ 0.00101 & -0.00120 & 0.00079 & -0.00153 & 0 \\ 0.00002 & -0.00064 & -0.00037 & 0.00023 & -0.00031 \\ -0.00327 & 0.00048 & 0.00031 & -0.00241 & -0.00156 \\ 0.00022 & -0.00015 & 0 & 0.00028 & -0.00084 \end{bmatrix}.$$

The long-run impact matrix is:

$$CB^{str} = \begin{bmatrix} 0.0064 & 0 & 0 & 0 & 0 \\ 0.0002 & 0.0009 & 0 & 0 & 0 \\ 0.0004 & 0.0015 & 0 & 0 & 0 \\ 0.0064 & 0 & 0 & 0 & 0 \\ 0.0011 & 0.0009 & 0 & 0 & 0 \end{bmatrix}$$

The solution of the control problem, derived in section 5.1 returns the following law for the instrument:

$$u_t = -FX_t + \nu_0 + \nu_1 t$$

where

$$F = \begin{bmatrix} -0.174 & -0.61 & -1.45 & -0.08 & 0.106 & -0.04 & 0.515 & 0.01 & -0.025 \end{bmatrix}$$

$$\nu_0 = -2.068 \text{ and } \nu_1 = -0.0010406.$$

The new VECM system, obtained by implementing the control rule above, has the following characteristics:

$$\alpha^* = \begin{bmatrix} -0.1298 & 0 & 0 & 0 \\ 0 & -0.4051 & 0.2325 & 0 \\ 0.0283 & 0 & 0 & -0.1939 \\ 0.1228 & -1.0580 & 0.6072 & -0.9860 \\ 0.0964 & -0.6585 & 1.1228 & -1.3350 \end{bmatrix} \quad \textit{and}$$

$$\beta^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -0.1412 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C^* = \begin{bmatrix} -0.1844 & -2.9610 & -5.7651 & 1.1337 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.1844 & -2.9610 & -5.7651 & 1.1337 & 0 \\ -0.0260 & -0.4181 & -0.8140 & 0.1601 & 0 \end{bmatrix}$$

How will the new long run impact matrix $C^* B^{*str}$ look like?

$$B^{str*} = \begin{bmatrix} -0.0022 & -0.0006 & 0.0009 & 0.0025 & 0 \\ 0.0010 & -0.0012 & 0.0008 & -0.0015 & 0 \\ 0 & -0.0006 & -0.0004 & 0.0002 & -0.0003 \\ 0.0033 & 0.0005 & 0.0003 & -0.0024 & -0.0016 \\ 0.0002 & -0.0019 & 0.0001 & -0.0001 & -0.0014 \end{bmatrix}$$

so $C^* B^{*str}$ results

$$C^* B^{*str} = \begin{bmatrix} -0.0064 & 0.0079 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.0064 & 0.0079 & 0 & 0 & 0 \\ 0.0009 & -0.0011 & 0 & 0 & 0 \end{bmatrix}$$

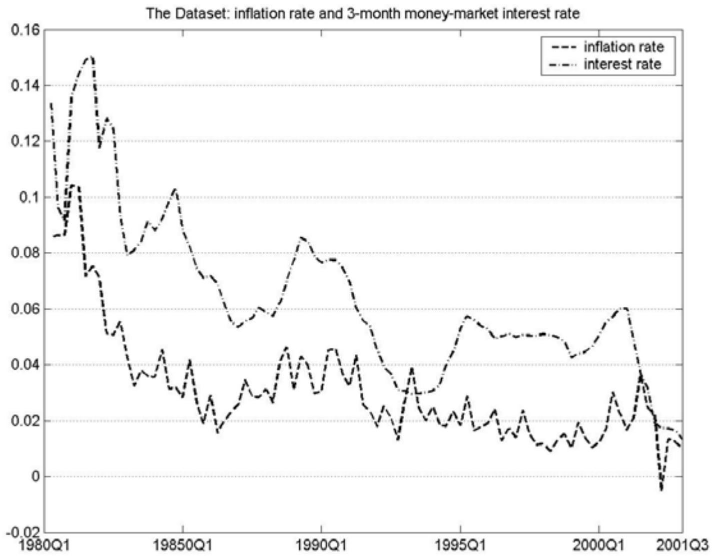

Figure 1: The Dataset: annualized inflation and the 3-month money market interest rate

Figure 2: The Dataset: log of real GDP in billions of 1996 dollars

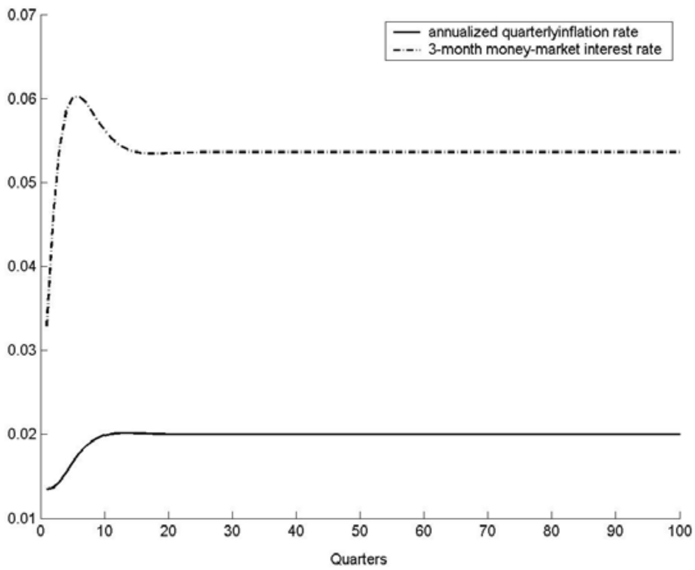

Figure 3: Deterministic Simulation of the controlled system (Model 1)

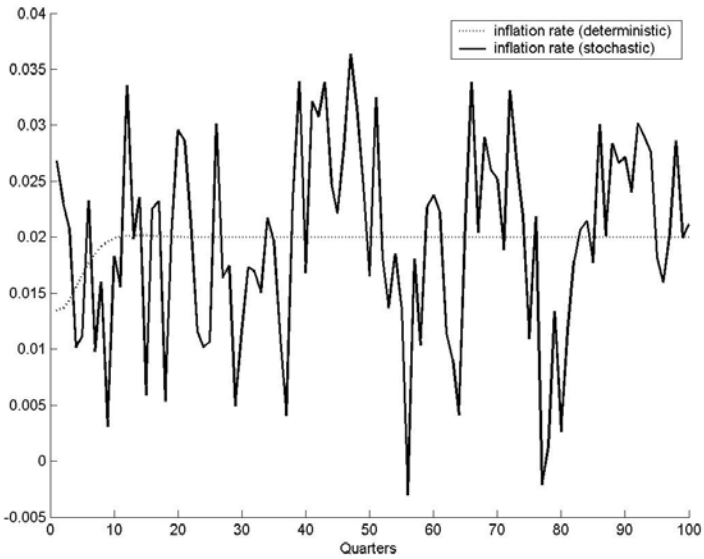

Figure 4: Deterministic and stochastic simulations of the target variable in the controlled system (Model 1)

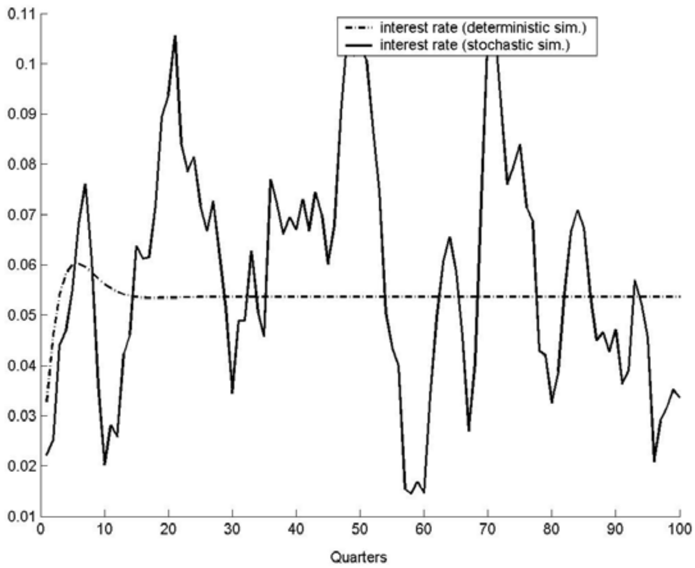

Figure 5: Deterministic and stochastic simulation of the policy instrument in the controlled system (Model 1)

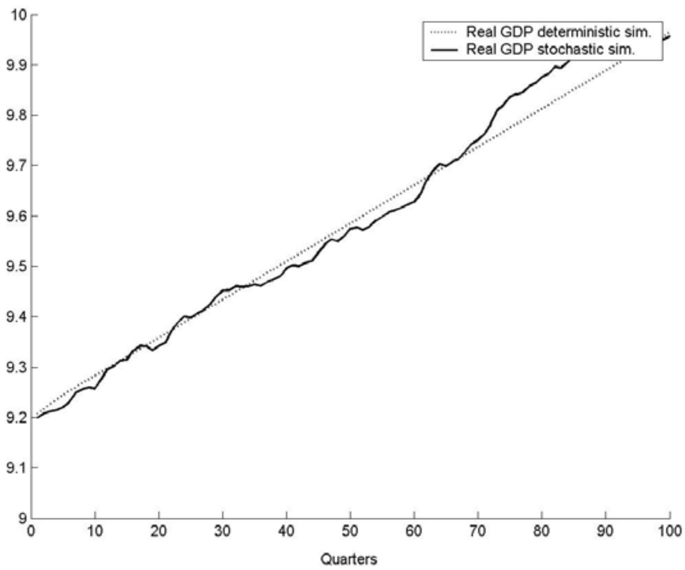

Figure 6: GDP in the simulated controlled system (Model 1)

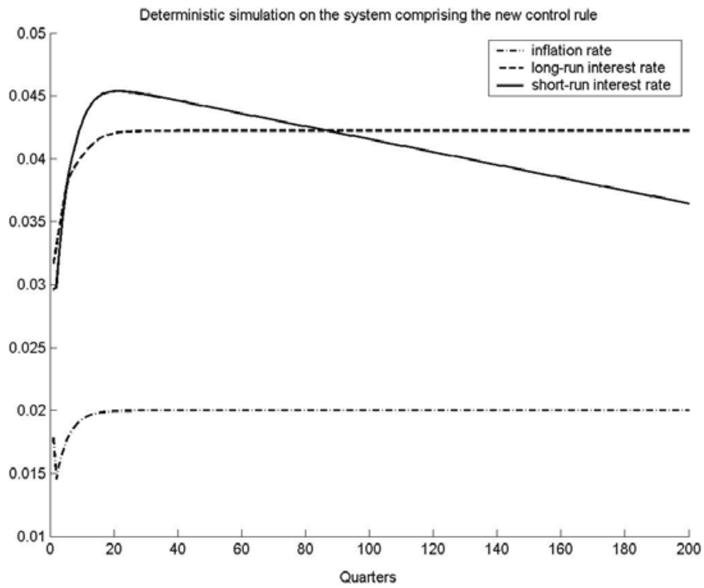

Figure 7: Deterministic simulation of the controlled model 2 (inflation, short-run and long-run interest rate)

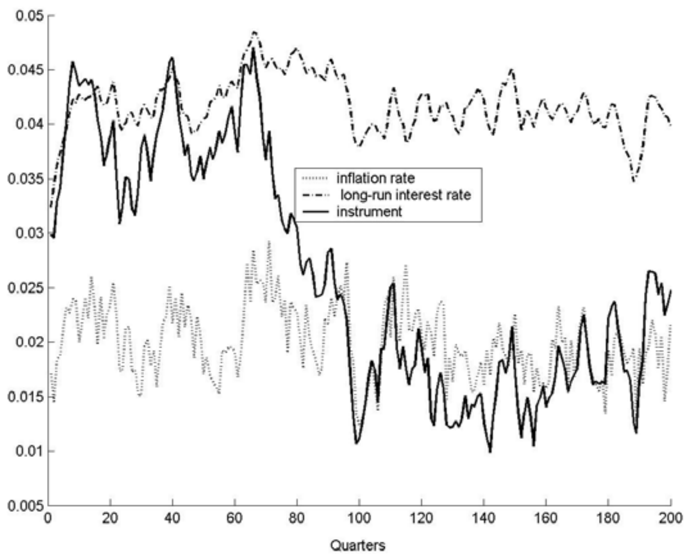

Figure 8: Stochastic simulation of the controlled 5-variable model 2 (inflation long-run and short-run interest rates)

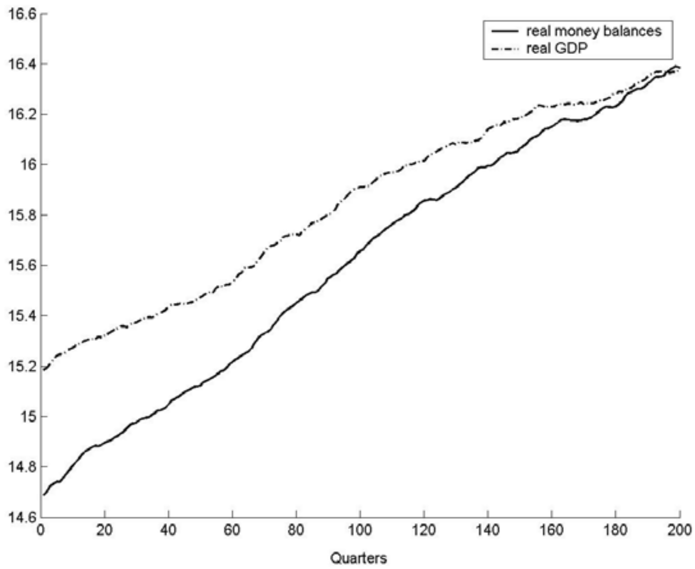

Figure 9: Stochastic simulation for the controlled model 2 (real money balances and real GDP)

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